An Ant on the Clock: Exponential Escape Bounds for Langton's Ant

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Abstract

We prove that the classical two-colour Langton ant exits every square of side n within $(13 \cdot 3^{n-1} - 2n^2 - 4n - 5)/2$ updates, uniformly over all initial colourings and internal poses. The proof uses an explicit integer-valued clock that increases by at least one on every update. Its finite range on a square gives an exponential escape bound with leading term $(13/6)3^n$. For a rectangle with N columns and fixed height H, an anisotropic rational potential gives an upper bound of order $N^{\lfloor H/2 \rfloor + 1}$. The integer clock also yields decidable complete-state reachability, two-sided orbit equivalence and a computable orbit normal form. Exact algebraic checks and finite-state audits accompany the proofs; local lemmas for the related supercritical rational clock are formalized in Lean.

1 Model and results

We seek a uniform bound on the time an ant can spend inside a finite region. A clock that gains at least one per update turns this temporal question into a spatial one: bound the range of its values inside the region.

A state is $X = (B, p, h)$, where $B \subset \mathbb{Z}^2$ is finite, $p \in \mathbb{Z}^2$, and $h \in \{N, E, S, W\}$. Headings are encoded cyclically by $0, 1, 2, 3$. The unit vectors are $e_N = (0, 1)$, $e_E = (1, 0)$, $e_S = (0, -1)$ and $e_W = (-1, 0)$. A state at time t is observed *before* update t. The update is

$$h' = h + 1 - 2\mathbf{1}_B(p) \pmod 4, \qquad B' = B \triangle \{p\}, \qquad p' = p + e_{h'}. \tag{1}$$

Write $F(X) = (B', p', h')$ for this update map. Thus white gives a right turn, black a left turn, and the read cell flips before the ant moves.

Let $S(N, H)$ be the maximum first-exit time from $Q_{N,H} = \{0, \dots, N-1\} \times \{0, \dots, H-1\}$ over all internal colourings, positions and headings. The exit move is counted. Exterior colours can be arbitrary: no exterior cell is read before the first exit.

Theorem 1.1. *For positive integers n, N, H,*

$$S(n, n) \le \frac{13 \cdot 3^{n-1} - 2n^2 - 4n - 5}{2}, \tag{2}$$

$$S(N, H) \le \left\lfloor \frac{9}{2}(NH + 16)(4N + 1)^{\lfloor H/2 \rfloor} \right\rfloor + 3. \tag{3}$$

In particular, $S(n, n) = O(3^n)$ and $S(N, H) = O_H(N^{\lfloor H/2 \rfloor + 1})$.

The square estimate improves the factorial upper bound of Etse [1] to exponential order in the side length. His fixed-height rectangle bound has degree $H - 1$ for even H and $H - 2$ for odd H; the degree above is smaller for every $H \ge 6$. For heights two and three, his linear

estimates remain sharper. The integer clock gives the leading square coefficient 13/6 through a unit-increment identity and an explicit finite range.

The common mechanism is positive clock growth within a finite range. For squares, absolute diagonal weights give an integer clock with unit minimum increment. For rectangles, anisotropic weights control the long direction while paired updates supply a uniform cost in the short direction. On the full finite-support state space, the same unit increment decides complete-state forward reachability and two-sided orbit equivalence. It also selects a unique, computable zero-crossing representative of each orbit (Theorems 6.1 and 6.3). These complete-state questions complement the circuit-prediction and infinite-background constructions of [2].

We construct the integer clock, sum its range on a square, and then derive the rectangular bound from a critical rational potential. The last part applies the same integer clock to complete-state orbit algorithms.

2 An integer clock

Define

$$\chi(x,y) = (-1)^{x+y}, \qquad \sigma(X) = \begin{cases} \chi(p), & h = N, S, \\ -\chi(p), & h = E, W, \end{cases}$$

and put $\varepsilon(h) = +1$ for $h = N, E$, and $\varepsilon(h) = -1$ for $h = S, W$. Each update changes both lattice parity and heading axis, so σ is invariant under F. Fix an integer centre c and set

$$s(q) = q_x + q_y - c, \qquad f(s) = 3^{|s|} - 1, \qquad J(s,e) = \begin{cases} f(s)/2, & es \geq 0, \\ -3f(s)/2, & es < 0. \end{cases}$$

The complete-state clock is

$$\Theta_c(X) = -\sigma(X) \sum_{q \in B} \chi(q) f(s(q)) + J(s(p), \varepsilon(h)). \tag{4}$$

Every $f(s)$ is even, so the finite sum defines a computable integer. The heading correction balances the current-cell flip, including at the centre where the absolute-value weight changes branch.

Theorem 2.1 (Integer increment)**.** *Let $e = \varepsilon(h)$, $e' = \varepsilon(h')$ and $s = s(p)$. Then*

$$\Theta_c(FX) - \Theta_c(X) = 1 + 4f(s)\mathbf{1}_{\{e' \neq e,\ es < 0\}} \geq 1. \tag{5}$$

Consequently $\Theta_c(F^t X) - \Theta_c(X) \geq t$ for every $t \in \mathbb{N}$.

Proof. Write $\delta = 1$ on a white read and $\delta = -1$ on a black read. The four headings and two colours in (1) give $\delta\sigma\chi(p) = ee'$ and $s(p') = s + e'$. The field increment is thus $-ee'f(s)$. Writing $J_\pm(s) = J(s, \pm 1)$, the defining powers give

$$\begin{aligned} J_+(s+1) - J_+(s) &= f(s) + 1, \\ J_-(s-1) - J_-(s) &= f(s) + 1, \\ J_-(s) - J_+(s) &= -2\,\mathrm{sgn}(s) f(s). \end{aligned}$$

For $e' = e$, the field term cancels $f(s)$, leaving one. For $e = 1, e' = -1$, the total is $1 + 2f(s) + J_-(s) - J_+(s)$, which is one for $s \geq 0$ and $1 + 4f(s)$ for $s < 0$. The other reversal follows with signs reversed. At $s = 0$, $f(s) = 0$, so both formulas agree. Summing proves the time estimate. □

3 Counting the clock range on a square

Fix a translated square $Q = \{x_0, \dots, x_0+n-1\} \times \{y_0, \dots, y_0+n-1\}$ and take $c = x_0+y_0+n-1$. Restrict the field sum in (4) to $B \cap Q$. This localized clock is finite for arbitrary exterior colouring, and (5) still holds between internal states: the only changed cell is inside Q.

Lemma 3.1 (Square range)**.** *For each fixed σ, the localized clock on internal states has range at most*

$$R_n = \frac{13 \cdot 3^{n-1} - 2n^2 - 4n - 7}{2}.$$

Proof. There are $n - |s|$ cells on diagonal s, for $|s| \le n-1$. The range of a binary weighted field sum is the sum of the absolute weights, which here is

$$\begin{aligned} S_n &= \sum_{s=-(n-1)}^{n-1} (n - |s|)(3^{|s|} - 1) \\ &= 2\sum_{j=1}^{n-1}(n-j)3^j + n - n^2 = \frac{3^{n+1} - 2n^2 - 4n - 3}{2}. \end{aligned}$$

For the last identity, write the weighted sum as $\sum_{k=1}^{n-1}\sum_{j=1}^{k} 3^j$ and sum the geometric series. Set $M = 3^{n-1} - 1$. All internal head corrections belong to $[-3M/2, M/2]$, an interval of length $2M$. Adding this to S_n gives R_n. □

Proof of (2). An infinite internal trajectory would increase the bounded localized clock by at least one at every update, which is impossible. Let T be first exit. The states at times $0, \dots, T-1$ are internal, so precisely $T-1$ updates join internal states. Telescoping (5) gives $T - 1 \le R_n$, proving the bound including the exit move. For $n = 1$ the result is $T \le 1$. □

4 The critical four-quadrant potential

For a long rectangle, the diagonal weights still grow exponentially with the long side. We instead assign separate growth rates to the two coordinates, keeping horizontal weights bounded as the length grows. The resulting potential supplies the rectangular bound through pairs of updates.

Use the same checkerboard invariant σ as above. Fix $a, b > 1$ and a centre $c = (c_x, c_y) \in \mathbb{R}^2$, and put $U(q) = a^{q_x - c_x}$, $V(q) = b^{q_y - c_y}$. Define

$$\Psi_{a,b,c}(X) = -\sigma(X) \sum_{q \in B} \chi(q)(U(q) - U(q)^{-1})(V(q) - V(q)^{-1}) + G_h(U(p), V(p)), \qquad (6)$$

where

$$\begin{aligned} G_N(U,V) &= \frac{2}{b+1}(U + U^{-1})(V - bV^{-1}), \\ G_E(U,V) &= \frac{2}{a+1}(U - aU^{-1})(V + V^{-1}), \\ G_S(U,V) &= \frac{2}{b+1}(U + U^{-1})(V^{-1} - bV), \\ G_W(U,V) &= \frac{2}{a+1}(U^{-1} - aU)(V + V^{-1}). \end{aligned}$$

This finite sum keeps the entire black field, including old debris. It is rational for rational a, b and integer c.

Lemma 4.1 (Critical identity)**.** *Suppose* $(a-1)(b-1)=4$*. At a state with* $U=U(p)$ *and* $V=V(p)$*, the increment* $\Psi_{a,b,c}(FX)-\Psi_{a,b,c}(X)$ *is given by Table 1. In particular it is strictly positive for every complete finite-support state.*

Heading before update	White/right	Black/left
N	$4U/V$	$4/(UV)$
E	$4/(UV)$	$4V/U$
S	$4V/U$	$4UV$
W	$4UV$	$4U/V$

Table 1: Exact increments on the critical curve.

Proof. Let $\delta=1$ on white and -1 on black, and let $\eta_h=1$ on vertical headings and -1 on horizontal headings. Only the current colour changes, so the field contribution is $-\eta_h\delta(U-U^{-1})(V-V^{-1})$. If (d_x,d_y) is the outgoing unit vector and h' the new heading, add $G_{h'}(Ua^{d_x},Vb^{d_y})-G_h(U,V)$. In each of the eight cases, its difference from the stated table entry simplifies to

$$\frac{(U^2+1)(V^2+1)(ab-a-b-3)}{UV(a+1)(b+1)}.$$

The numerator vanishes on the critical curve. □

Every entry in the increment table is positive. Its dependence on position allows an individual increment to be arbitrarily small. For rectangles, pairing consecutive updates gives a uniform lower bound on their combined cost in the short direction.

5 Pairing in the short direction

For first exit from $Q_{N,H}$, restrict the field sum in (6) to $B\cap Q_{N,H}$. The critical increment table continues to hold between internal states, since each flipped cell lies inside the rectangle. This finite localization covers arbitrary exterior colourings, just as for the integer clock on a square.

For $Q_{N,H}$ choose

$$a=1+1/N,\qquad b=1+4N,\qquad c=(0,\lfloor(H-1)/2\rfloor),\qquad k=\lfloor H/2\rfloor.$$

Then $(a-1)(b-1)=4$, $|y-c_y|\le k$, and $1\le U\le a^{N-1}<3$. For the last inequality, the binomial expansion bounds $(1+1/N)^N$ by $\sum_{j=0}^{N}1/j!<3$.

Lemma 5.1. *While the head remains in* $Q_{N,H}$*, a valid range bound is*

$$R_{N,H}=3(NH+16)b^k.$$

A two-update segment beginning with a horizontal heading, whose two read positions are internal, increases the potential by at least

$$\delta_N=4a^{-(N-1)}(1+1/b)>4/3.$$

Proof. Each field coefficient has magnitude at most $3b^k$, since $|U-U^{-1}|<3$ and $|V-V^{-1}|\le b^k$. The field range is therefore at most $3NHb^k$. Each G_h is a sum of four monomials with coefficient magnitudes at most 2; each monomial has magnitude at most $3b^k$. Hence $|G_h|\le 24b^k$, and the head range is at most $48b^k$.

The first move in the asserted pair is vertical, say by $v \in \{-1, 1\}$. The two entries of Table 1 have vertical factors b^j and b^{-j-1}, where $j = v(y - c_y)$ is an integer. Both are read at the same x, so both horizontal factors are at least $a^{-(N-1)}$. The elementary integer inequality $b^j + b^{-j-1} \geq 1 + 1/b$ proves the asserted cost. It is greater than $4/3$ because $a^{N-1} < 3$. □

Proof of (3). The arrival axes alternate. The internal-to-internal updates can thus be grouped into the pairs of Lemma 5.1, omitting at most one initial and one final update. With r pairs, $T \leq 2r + 3$ and $r\delta_N \leq R_{N,H}$. An indefinitely confined trajectory contradicts the same finite range. Consequently

$$T \leq \left\lfloor \frac{2R_{N,H}}{\delta_N} \right\rfloor + 3 \leq \left\lfloor \frac{9}{2}(NH + 16)(4N + 1)^k \right\rfloor + 3.$$

□

The proof also gives a sharper computable bound by replacing $R_{N,H}$ with

$$\widehat{R}_{N,H} = \sum_{q \in Q_{N,H}} |(U(q) - U(q)^{-1})(V(q) - V(q)^{-1})| + \max_{q,h} G_h(q) - \min_{q,h} G_h(q).$$

The bound $\lfloor 2\widehat{R}_{N,H}/\delta_N \rfloor + 3$ uses only finite rational arithmetic. It is not the exact maximum first-exit time.

By a quarter-turn rotation one can exchange N and H and take the smaller of the two bounds. Etse's Theorem 12 [1], for $N \geq H \geq 2$, has polynomial degrees $H - 1$ and $H - 2$ in the even and odd cases respectively. The degree $\lfloor H/2 \rfloor + 1$ is strictly smaller for $H \geq 6$.

6 Complete-state orbit algorithms

Fix the single integer clock $\Theta = \Theta_0$ throughout this section. Here the comparison is between complete states, so the same fixed clock is used for every input. Its uniform increment converts a target clock value into a finite simulation budget.

Theorem 6.1 (Forward reachability)**.** *Given complete finite-support states X, Y, it is decidable whether $F^t X = Y$ for some $t \geq 0$. If such a time exists it is unique and satisfies*

$$t \leq N(X, Y) := \Theta(Y) - \Theta(X).$$

Proof. Different σ values or a negative clock difference rule out a match. Otherwise compute the displayed integer and simulate through times 0 to $N(X, Y)$, inclusive, comparing the entire black set, position and heading. The time bound in Theorem 2.1 rules out every later match. Strict increase also rules out two different matching times. Each state at finite time is finite, and all integer arithmetic and full-state comparisons are effective. □

The inverse update is equally explicit. Given (B, p, h), put $q = p - e_h$. If $q \in B$, the preceding update was white-to-black, so the old heading was $h - 1$. If $q \notin B$, it was black-to-white, so the old heading was $h + 1$. In either case flip q and put the predecessor at q. This defines a computable bijection on the set $\mathcal{X}$ of complete finite-support states.

Corollary 6.2 (Orbit equivalence)**.** *The relation $\exists t \in \mathbb{Z}\ F^t X = Y$ is decidable, with a computable signed time offset whenever it holds.*

Proof. If the states differ, their time order on an orbit must agree with the strict order of their clock values. Apply Theorem 6.1 in that order; equal clock values allow a match only for identical states. □

Theorem 6.3 (Computable orbit section)**.** *Define the decidable set*

$$\mathcal{S} = \{C \in \mathcal{X} : \Theta(C) \geq 0 > \Theta(F^{-1}C)\}.$$

Every two-sided orbit meets $\mathcal{S}$ exactly once. Its representative $C(X)$ and the unique integer $k(X)$ with $X = F^{k(X)}C(X)$ are computable. The map $X \mapsto (C(X), k(X))$ is a computable bijection between $\mathcal{X}$ and $\mathcal{S} \times \mathbb{Z}$ under which F acts by $(C, k) \mapsto (C, k+1)$.

Proof. Forward values tend to $+\infty$ by Theorem 2.1. Applying the same inequality to $F^{-t}X$ shows that backward values tend to $-\infty$. Strict increase therefore yields exactly one crossing from negative to nonnegative values. Search forward when $\Theta(X) < 0$ and backward otherwise, counting the signed number of moves. A coarse bound for the moves and final predecessor check is $|\Theta(X)| + 1$. The inverse of the displayed correspondence computes $F^k C$ using forward or inverse updates as appropriate. □

The normal form separates the position along an orbit from its orbit label $C \in \mathcal{S}$. A row-parity invariant shows that infinitely many distinct labels are needed.

Lemma 6.4 (Infinitely many orbit representatives)**.** *The section $\mathcal{S}$ is infinite.*

Proof. For each integer row j, the bit

$$I_j(X) = |B \cap (\mathbb{Z} \times \{j\})| + \mathbf{1}_{\{p_y = j,\ h \in \{E,W\}\}} \pmod 2$$

is invariant. On a vertical-to-horizontal turn the new head stays in the read row and its indicator cancels that row's colour flip. On a horizontal-to-vertical turn the old head indicator cancels the flip. All other rows are unchanged. States with the head at the origin facing north and sole black cell $(0, j)$ have distinct invariant vectors $(I_i)_{i \in \mathbb{Z}}$. They therefore lie on distinct orbits, each with its own section representative. □

Computational cost. The clock gives explicit finite simulation budgets for the orbit algorithms. With sparse binary coordinates, exact arithmetic may require exponential space in the coordinate bit length: the expanded integer representation of $3^{|x|}$ has length proportional to $|x|$. The algorithms establish decidability with this representation-dependent cost.

7 Verification and reproducibility

The integer-clock artifact checks every colour/pose state in squares of sides one through three: 18,696 initial states, 12,416 internal transitions and exact maximum exit times $1, 6, 17$. It checks the closed-form field range for 100 sizes. A separate algebraic check verifies all sign branches of the integer increment symbolically, including the central diagonal. These finite controls support the implementation; the range argument and telescoping proof establish the result for every side length and every exterior colouring.

For the critical potential, the retained checker exercises 3,000 local identities, 6,400 four-step cases, 10,608 horizontal-arrival pairs and 144 rectangular ranges. Its separate bit simulation exhausts 28,808 initial states over eight small rectangles; 80 additional trajectories retain exterior debris. An independent implementation checks the eight cleared-denominator identities as Laurent polynomials in a, b, U, V, together with short-block identities and exit-step rounding. The written geometric range and pairing arguments supply the universal rectangular bound.

The artifact includes the supercritical rational-clock checks and an independent functional-graph audit of all 4,213,000 colour/pose states in squares through side four, whose maximum exit times are $1, 6, 17, 46$. Lean 4.30.0 checks the local algebra and telescoping lemmas for the supercritical clock in the appendix; its printed axioms are `propext`, `Classical.choice` and `Quot.sound`. The integer and critical-potential proofs are analytic, with exact algebraic checks, and are separate from those Lean lemmas.

8 Limitations

The escape estimates are uniform upper bounds. Their asymptotic sharpness, including the exponential growth rate for squares, remains open. The orbit algorithms take the entire finite black field and pose as input and compare complete states; deciding arbitrary local-observation events or eventual highway behaviour requires additional arguments. Their explicit simulation budgets establish decidability with the representation-dependent cost described above. The formalized results cover the supercritical local identities and telescoping; the integer and critical escape bounds use the analytic proofs and exact algebraic checks given here.

9 Conclusion

An absolute-diagonal integer clock converts finite spatial range into a unit budget for ant updates. Summing its weights gives an explicit exponential square-escape bound, while anisotropic critical weights give polynomial bounds for fixed-height rectangles. On complete finite-support states, the same integer clock provides a finite target-search budget and a unique zero-crossing representative of every two-sided orbit. These results connect the local turn-and-flip rule to quantitative escape and effective orbit structure.

Source and software availability. The computational supplement is fixed at repository commit `e932522`, clock project. Its `sources.json` gives the verification-source hashes; `evidence/integer/` and `evidence/critical/` contain the exact reports for the two escape arguments. The accompanying source archive includes these checkers and reports together with this article's source. From the repository root, run:

```
python3 clock_escape/reproduce.py --integer-only
python3 clock_escape/reproduce.py --critical-only
python3 clock_escape/reproduce.py
make -C clock_escape/paper
```

The algebraic checks use Python's standard library; the formalized lemmas use Lean 4.30.0.

AI assistance. Generative AI assisted with developing and checking arguments, implementing verification programs, and revising the exposition.

A The supercritical rational clock

The supercritical rational construction supplies the local identities formalized in Lean and a smooth family of strictly increasing potentials. Use the same σ and ε as in Section 2. Fix $c \in \mathbb{Z}$ and a rational $r > 3$. The parameter c centres the diagonal coordinate, while r controls the growth of the weights. Write $s(q) = q_x + q_y - c$ and set

$$f_r(s) = r^s + r^{-s}, \tag{7}$$

$$g_r(s, +1) = \frac{2}{r+1}(r^s - r^{1-s}), \tag{8}$$

$$g_r(s, -1) = \frac{2}{r+1}(r^{-s} - r^{s+1}), \tag{9}$$

$$\Phi_{r,c}(X) = -\sigma(X) \sum_{q \in B} \chi(q) f_r(s(q)) + g_r(s(p), \varepsilon(h)). \tag{10}$$

Finite support makes (10) a computable rational number, including when coordinates are negative.

Lemma A.1 (Increment identity)**.** *Let* $U = r^{s(p)}$, $V = r^{-s(p)}$, *and*

$$d_r = \frac{r-3}{r+1}, \qquad E_r = \frac{5r+1}{r+1}.$$

If $e = \varepsilon(h)$ *and* $e' = \varepsilon(h')$, *then*

(e, e')	$\Phi_{r,c}(FX) - \Phi_{r,c}(X)$
$(+,+)$ *or* $(-,-)$	$d_r(U+V)$
$(+,-)$	$d_rU + E_rV$
$(-,+)$	$E_rU + d_rV$

In particular $\Phi_{r,c}(FX) - \Phi_{r,c}(X) \geq 2d_r > 0$.

Proof. Let $\delta = +1$ on a white read and $\delta = -1$ on a black read. From the four headings and two colours in (1),

$$\delta\sigma(X)\chi(p) = ee', \qquad s(p') = s(p) + e'.$$

The black-field contribution to the increment is therefore $-ee'(U+V)$. Subtracting the old pose term from the new one gives

$$-ee'(U+V) + g_r(s(p)+e', e') - g_r(s(p), e).$$

Substitution of the two definitions of g_r gives the displayed table. For example, for $(e, e') = (+,+)$, the pose increment is $2(r-1)(U+V)/(r+1)$, which yields $d_r(U+V)$ after the field increment. For completeness, after multiplying by $r+1$, the other three cases are

$$\begin{aligned}
(-,-): &\quad -(r+1)(U+V) + 2(rV-U) - 2(V-rU),\\
(+,-): &\quad (r+1)(U+V) + 2(rV-U) - 2(U-rV),\\
(-,+): &\quad (r+1)(U+V) + 2(rU-V) - 2(V-rU).
\end{aligned}$$

Expansion gives the corresponding polynomial identities. Since $UV = 1$, $U + V \geq 2$; also $E_r \geq d_r > 0$. These give the lower bound in every case. □

Choosing $r = 4$ and $c = 0$ gives a second uniform time estimate.

Theorem A.2 (Strict clock)**.** *The function* $\Phi = \Phi_{4,0}$ *is computable on every complete finite-support state and satisfies*

$$\Phi(F^tX) - \Phi(X) \geq \frac{2}{5}t \qquad (t \in \mathbb{N}). \tag{11}$$

Proof. Apply Lemma A.1 with $r = 4$ and sum the increments. □

References

[1] Kossi Roland Etse. How Long Can The Escaping Ant Be Confined? In *51st International Symposium on Mathematical Foundations of Computer Science (MFCS 2026)*, LIPIcs 386, 49:1–49:17, 2026. doi:10.4230/LIPIcs.MFCS.2026.49.

[2] Anahí Gajardo, Andrés Moreira and Eric Goles. Complexity of Langton's ant. *Discrete Applied Mathematics* 117:41–50, 2002. doi:10.1016/S0166-218X(00)00334-6.