Counting Defects: Semilinear Reachability and Resource Bounds for Langton's Ant

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Abstract

Fix a number of initially black cells and a target number of departures from the standard Langton highway. Which placements of those cells make the ant reach that target? We count a departure whenever ten consecutive turns form a word absent from the standard 104-turn cycle. For each supplied pair of integers m, N, we prove that the placements of exactly m black cells producing at least N such departures form an effectively computable semilinear set: a finite union of integer translates of finitely generated additive monoids. Initial coordinates are unrestricted. The construction also describes the exact time, pose and entire black field just after the Nth departure. The key reduction represents the intervening legal motion by a finite collection of words with variable repetition counts. Consecutive visits to each cell enforce the ant's read-and-flip rule throughout these compressed traces. This yields decidable synthesis under integer-linear coordinate constraints and piecewise rational-affine resource formulas for supplied word families. Positive instances have polynomial-bit witnesses when m, N are written in unary. A restricted quantifier-free encoding makes the spatial read/write tests polynomial in the supplied description size. Together these results turn finite departure targets into exact arithmetic questions about complete ant states.

1 Problem and main result

Suppose we may place m black cells anywhere on an otherwise white plane. We want the ant to depart from the local turn patterns of its standard highway at least N times. The cell coordinates can be arbitrarily large, and the Nth departure may occur after an arbitrarily long flight. This is therefore a synthesis problem over unbounded coordinates and time. We show that each fixed target m, N has an exact description in integer-linear arithmetic.

Here *Presburger arithmetic* means first-order arithmetic with integer addition, order and fixed integer constants, but without multiplication of two variables. Its definable sets are exactly the semilinear sets: finite unions of sets $a + \mathbb{N}v_1 + \cdots + \mathbb{N}v_r$ with integer vectors a, v_i. Such a description allows one to test whether any initial placement meets additional coordinate constraints. Our construction also records the complete state when the target is first reached. A state is (B, p, h), where $B \subset \mathbb{Z}^2$ is the *complete* finite set of black cells, p is the head position, and $h \in \{N, E, S, W\}$. A white read turns right, a black read turns left, the current cell flips, and the ant then moves one unit. Time t denotes the state immediately before update t.

Record a right turn by R and a left turn by L. Let W be the standard period-104 turn word printed in Appendix A, and let $\mathcal{L}_{10}$ be its cyclic ten-letter factors. A *contained defect* of a finite forward word is a length-ten consecutive subword outside $\mathcal{L}_{10}$. For example, a segment of the repeated highway word has no defects, even across the boundary between two copies. A contained defect is indexed by its final update $t \geq 9$; its post-event state is at time $t + 1$. The ten-factor attraction theorem states that a finite-support orbit is eventually standard if and only if it has finitely many such defects [5].

Normalize the head to the origin facing north. For fixed $m \geq 0$ and $N \geq 1$, let $E_{m,N}$ contain the ordered, pairwise-distinct tuples $z = (x_1, y_1, \ldots, x_m, y_m)$ for which the actual ant with exactly these black cells has at least N contained defects. Coordinates have no range restriction.

A finite-fibre field family is a definable set of pairs (z, q) such that, for each parameter tuple z, only finitely many lattice cells q belong to the field. Describing this membership relation specifies every endpoint cell, including unvisited initial inventory. For later resource questions, write

$$\mu(N) = \min\{|B| : \text{the complete finite input } B \text{ produces at least } N \text{ contained defects}\}.$$

The initial pose is normalized as above. This quantity asks for the least black inventory needed to reach a finite target.

Theorem 1.1 (All-coordinate fixed-budget synthesis)**.** *There is an effective construction, uniform in the supplied integers m, N, of a Presburger formula defining $E_{m,N}$. The exact time and pose immediately after the Nth defect have an effectively Presburger-definable graph. The complete black field at that endpoint is an effective finite-fibre Presburger family over the initial coordinates.*

In particular, existence, nonexistence, and intersection with a prescribed Presburger coordinate domain are decidable. A finite union handles "at most m black cells." This is not one fixed formula in which arbitrary m, N are additional free variables. The finite union can be large; its effective construction establishes decidability, while the complexity results below concern compressed positive witnesses and a restricted encoding.

The quantity N is a target for a finite prefix: the ant may have further defects after the reported endpoint. The formula describes inputs reaching the Nth defect and says nothing about a bound on the total future count.

The central difficulty is physical execution: the ant carries an unbounded mutable spatial store, so a short word description must retain every earlier read and every initially black cell. Consecutive same-cell relations solve this problem exactly. A finite ten-factor cover then yields the all-coordinate construction, and unary time-fibre counting yields its resource refinement. Sections 3–4 build the arithmetic description and prove that its finite family of word forms covers every Nth-event prefix. Sections 5 and 6 derive resource spectra and small witnesses; Section 7 gives the restricted quantifier-free encoding. A *word schema* means a fixed sequence of literal words and repeated words whose nonnegative repetition counts are parameters.

Symbolic feasibility for ant words has an established history [1]. The equivalence of Presburger descriptions and semilinear sets is classical [8]. Flat-system verification already provides run schemas, small-witness arguments and quantifier-free counting encodings [3, 7]. Our reduction supplies the additional physical semantics of the ant's complete mutable black field. The companion factor paper [5] supplies the bounded-fold exclusion and complete finite cover. Together these yield exact all-coordinate synthesis, ant-specific supply spectra and unary-budget NP membership. The restricted backend encodes spatial read/write parity as well as factor counts in polynomial size. Unary counting and piecewise-affine Presburger functions are established tools [9, 10]; the explicit fold–consumer spectrum is an application with sharp ant-specific constants.

2 Consecutive reads without a parity-counting oracle

Fix a parameter tuple θ in a Presburger domain D. Suppose a prescribed kinematic trace is represented by a Presburger function V_θ from a contiguous interval of update times to physical cells. Its domain is either $\{0, \dots, T(\theta) - 1\}$ or $\mathbb{N}$. Let R_θ, L_θ partition its times according to the requested letters. The movements and headings are assumed to satisfy the literal turn rule; Section 3 constructs such traces.

Define

$$C_\theta = \{(s, t) : s < t,\ V_\theta(s) = V_\theta(t)\}.$$

Relational composition uses the same parameter tuple throughout.

Lemma 2.1 (Successor subtraction)**.** *The relation*

$$\mathrm{Next}_\theta = C_\theta \setminus (C_\theta \circ C_\theta) \tag{1}$$

consists exactly of consecutive visits to the same physical cell.

Proof. A pair in C_θ fails to be consecutive precisely when a read time u strictly between its endpoints visits their common cell. This is exactly membership in $C_\theta \circ C_\theta$. No transitive closure is computed. □

Put $\mathcal{T}_\theta = \operatorname{dom} V_\theta$ and

$$\mathrm{First}_\theta = \mathcal{T}_\theta \setminus \operatorname{ran} C_\theta.$$

For a complete initial black predicate $B(\theta, x, y)$, write $\mathrm{Hits}_\theta = \{t : B(\theta, V_\theta(t))\}$.

Theorem 2.2 (Exact realization)**.** *The parameter assignments realizing the prescribed trace are exactly those satisfying*

$$\mathrm{Next}_\theta \cap ((R_\theta \times R_\theta) \cup (L_\theta \times L_\theta)) = \varnothing, \tag{2}$$

$$\mathrm{First}_\theta \cap L_\theta = \mathrm{First}_\theta \cap \mathrm{Hits}_\theta\,. \tag{3}$$

Their set is effectively Presburger-definable.

Proof. Both conditions are necessary by flipping and initial colour. For sufficiency, induct in ordinary time. A first visit has the right initial colour by (3). Otherwise a nearest preceding visit exists: a given time has only finitely many predecessors, even for an infinite trace. Its letter is opposite by (2), so the current physical colour is correct. The requested movement is therefore the actual movement. All integer times are covered. Projection of violations and complementation in D give the Presburger parameter set. □

This quantified form of the established same-cell alternation criterion [1] handles all parameter assignments together. Its advantage is that it avoids assuming an arbitrary definable cardinality has a definable parity.

Without a supplied field, call a trace *feasible* when it satisfies the same-cell alternation condition (2); it is then realizable from some initial colouring. For a finite feasible trace, the smallest supporting initial black set is

$$B_{\min}(\theta) = V_\theta(\mathrm{First}_\theta \cap L_\theta). \tag{4}$$

For an infinite trace this set may be infinite. Existence on an unspecified colouring must not be mistaken for existence on a finite support.

Theorem 2.3 (Complete endpoint)**.** *For a finite trace, define* $\mathrm{Last}_\theta = \mathcal{T}_\theta \setminus \operatorname{dom} C_\theta$*. On its realization domain the final black field is*

$$B_{\mathrm{out}}(\theta) = (B(\theta) \setminus V_\theta(\mathcal{T}_\theta)) \cup V_\theta(\mathrm{Last}_\theta \cap R_\theta). \tag{5}$$

It is an effective finite-fibre Presburger family when B is one.

Proof. Unvisited initial cells retain their colour. A last right turn leaves black; a last left turn leaves white. These cases are disjoint and exhaustive. Only finitely many updates occur at each parameter assignment. □

Equation (5) retains every unread initial cell.

3 Unbounded powers and effective resource spectra

Consider a finite bounded-power schema

$$v_0 u_1^{k_1} v_1 \cdots u_s^{k_s} v_s, \qquad k_i \geq 0, \tag{6}$$

with fixed words and heading-restoring u_i. Fixed pieces may rotate heading. An occurrence at phase r and copy index j has time and position of the form

$$t = b_r + c_r k + \ell_r j, \qquad q = a_r + A_r k + d_r j, \qquad 0 \leq j < k_i. \tag{7}$$

The coefficients are fixed integers. There are no products of two unbounded variables. Zero powers and affine reparameterizations are allowed. A final infinite power simply replaces its copy bound by $j \geq 0$.

Theorem 2.2 thus extends fixed-word replay to arbitrarily long parameterized operations, and Theorem 2.3 preserves the complete field. A variable rotating power can first be split by its exponent modulo its rotation order; our implementation instead requires this preprocessing explicitly. Finite-fibre initial Presburger families may include finite unions or XORs of arithmetic segments with fixed strides and variable finite lengths.

Defect end-times in a schema are Presburger: intersect shifted letter-time sets for each allowed factor and subtract their union from eligible times. This treats cross-block windows and zero powers exactly.

The following standard counting fact combines unary counting elimination [9, Theorem 5.4] with Presburger cell decomposition [10, Theorem 1]. We give the elementary effective argument needed here.

Lemma 3.1 (Unary finite-fibre counting)**.** *For an effectively Presburger set $F \subseteq \mathbb{Z}^d \times \mathbb{Z}$ with finite fibres, $\theta \mapsto |F_\theta|$ is effectively rational-affine on the cells of a finite Presburger partition.*

Proof. Apply Presburger quantifier elimination [2]. Split parameter residues and the counted variable modulo a common multiple of the constant moduli. Writing $t = r + Qz$, each remaining atom either constrains the parameters alone or supplies a floor/ceiling affine bound on z. Use a disjoint Boolean partition, then partition parameters by the ordering of the finitely many bounds and their constant-denominator residues. Every nonempty counted cell is an integer interval with two finite rational-affine endpoints. Summing disjoint interval lengths proves the claim. Finiteness rules out unbounded intervals on the applicable parameter cell. □

Theorem 3.2 (Initial-black and defect spectrum)**.** *On the feasible domain of* (6)*, the minimum initial-black cost $c(\theta) = |\operatorname{First}_\theta \cap L_\theta|$ and the contained-defect count $d(\theta)$ have effective semilinear graphs and are piecewise rational-affine.*

Proof. Apply Lemma 3.1 to the two unary sets of times. Different first-read times refer to different cells, so (4) identifies the first count with initial-black cost. □

This does not assert affine counting for arbitrary planar Presburger fibres; for instance an n by n square has n^2 cells. The counted objects here are one-dimensional time sets. The runtime implements exact time sets and fixed integer thresholds via

$$|F_\theta| > b \iff \exists t_0 < \cdots < t_b \quad \bigwedge_i t_i \in F_\theta.$$

The counting lemma gives a constructive partition algorithm; the exact integer-threshold tests above suffice for the supplied-schema queries.

For any supplied finite schema collection and fixed mass bound, unboundedness of its defect spectrum is consequently decidable. If unbounded, a semilinear component gives an affine family of finite witnesses. Such a parameter family is not a fixed infinite counterexample.

4 A finite cover of all prefixes reaching a fixed defect count

The ant-specific bridge is the existing ten-factor macro theorem [5]. We state its certificate content to make the dependency explicit. The vertices are feasible 149-letter words whose every contained ten-factor is legal; an edge appends one letter and discards the first, provided the resulting 150-letter word is feasible and every ten-factor is legal. The graph has 6,501 vertices and 6,605 edges. Its cyclic components have 104, 24 and 366 vertices. The fixed words are printed in Appendix A. The 366-state component maps to a graph with two loops: a translating word A of length 28 and a heading-reversing word B of length 102. The established physical obstruction is

$$BA^nBA^mB \text{ is infeasible for every } n, m \geq 0. \tag{8}$$

It is an all-parameter same-cell conflict, not an observation at small lengths.

Counting at most two completed long loops and a third incomplete loop yields a finite covering lift with 7,595 vertices and 7,763 edges. Its only cyclic components are simple cycles of lengths $24, 28, 28, 28, 104$. The acyclic condensation supplies bounds of five cycle stages, 956 acyclic updates, and 212 for the sum of visited cycle lengths. Our independent reconstruction regenerates every feasible word, checks every quotient edge, rebuilds the lift and recovers these bounds. The exact arithmetic kernel separately verifies (8) over the entire quadrant.

The graph construction proves the following precise coverage statement.

Proposition 4.1 (Finite legal-word cover, from [5])**.** *Every physically feasible finite word with no contained defect has a representation $v_0u_1^{k_1}v_1\cdots u_r^{k_r}v_r$ with $r \leq 5$, $k_i \geq 0$, and $\sum_i |v_i| \leq 1317$. Each u_i is a cyclic phase of W, A or T. These words restore heading. From a north-facing pose the displayed representatives have displacements $(-2,-2), (0,-4)$ and $(0,-2)$, respectively; cyclic phases have quarter-turn rotations of those vectors.*

A finite defect-free word uses at most 149 initial letters before this graph cover applies. Every cyclic visit is a power followed by a partial cycle and possible exit. Therefore it has a schema with at most five powers and at most $149 + 956 + 212 = 1317$ nonpowered letters. The number of ordinary updates is not bounded: all exponents remain free.

Remove the last letter of each of the first N defect windows from the prefix ending at the Nth defect. The at most $N + 1$ remaining contiguous pieces contain no defect: any bad window wholly in a piece would have had its last letter removed. Apply Proposition 4.1 to each piece and restore the N removed letters as literal separators. Using the relaxed constant $2048 \geq 1317$ gives a finite schema family with

$$s \leq 5(N+1), \qquad \sum_i |v_i| \leq 2048(N+1) + N. \tag{9}$$

All power words are cyclic phases of W, A, T. This overapproximation may be redundant. Exact arithmetic semantics and exact defect count are subsequently imposed, so no nonphysical word is accepted.

Proof of Theorem 1.1. For each schema in (9), introduce its exponent variables, make the initial field exactly the distinct coordinate list z, and conjoin the realization predicate, exactly N contained defects, and a defect at the last letter. Existentially project exponents and take the finite union. Every genuine Nth-event prefix is covered; every satisfying assignment is a genuine prefix by Theorem 2.2. Retain the affine time and pose to obtain their graph. Formula (5) gives the black field; different schemas representing the same prefix agree by determinism. □

The finite union supplies a total mathematical decision procedure; all remaining parameters are eliminated by exact integer arithmetic.

5 Supply budgets for a finite schema collection

A supplied finite collection fixes its word descriptions while their exponents remain free. Once its resource image is effectively semilinear, the following dichotomy is a direct consequence of the signs of its period generators. The ant-specific work is constructing the complete physical resource image; the generator argument applies to any semilinear subset of $\mathbb{N}^2$.

Theorem 5.1 (Supply dichotomy)**.** *For every finite collection of bounded-power schemas, exactly one of the following alternatives holds, and the alternative is effectively decidable:*

1. *Some fixed initial cost c_0 supports arbitrarily many contained defects. One can extract a schema and an affine ray of exponents $k(n) = k_0 + nv$ with $c(k(n)) = c_0$ and $d(k(n)) = d_0 + n\delta$, $\delta > 0$.*
2. *There are computable rational $\alpha \geq 0, \beta$ such that $d(w) \leq \alpha c(w) + \beta$ for every feasible instance in the collection.*

Proof. Project the semilinear resource graph to $(c, d) \in \mathbb{N}^2$ and compute a finite union of linear sets $b + \sum_j \mathbb{N} g_j$. A generator with zero cost coordinate and positive defect coordinate gives the first alternative. Conversely, if cost stays bounded while defects grow, positive-cost generators have bounded multiplicities in some component, so a positive vertical generator is necessary. When there is none, set

$$\alpha = \max(\{0\} \cup \{g_{j,d}/g_{j,c} : g_{j,c} > 0\}), \qquad \beta = \max_b (b_d - \alpha b_c).$$

The inequality holds at each base and is preserved by every generator. Use zero constants for the empty spectrum. To extract the first alternative's exponents, fix its c_0 in the full (k, c, d) graph. An unbounded component has a generator with positive d coordinate, giving the asserted affine ray. The minimal seeds are computed by first-left reads for each parameter. □

The ray describes a family of finite inputs. The constants in the other alternative belong to the specified collection. When a collection has unbounded defects and no vertical ray, its least supply for at least N defects is $\Theta(N)$: the affine budget gives the lower bound, and any positive-defect generator gives the upper bound.

5.1 A two-fold warm-up

The arithmetic construction gives the following exact family. Using the words of Appendix A, put $A = \mathrm{L}B_{75} \cdots B_{101}$ with $|B| = 102$. All word indices start at zero.

Proposition 5.2. *The finite word BA^nBA^m is realizable from some finite support exactly when $n \geq 1$ and $0 \leq m < n$. Its minimum initial-black cardinality is*

$$c(n, m) = 5n + 5m + 47.$$

The feasible domain is computed using (2). Its equality with the displayed integer region is checked by exact Presburger set operations. For the cost, the first B supplies 24 distinct first-left reads, the second supplies 23, and every copy in either A run supplies exactly the five local indices $0, 9, 15, 26, 27$. The full time-set identity is checked parametrically, not extrapolated from sample values. Its disjoint blocks give the formula. Here is the complete first-left table. A row gives global times by adding its offset to its local indices; each j ranges over integers.

block	offset	local indices
first B	0	F_0
first A run	$102 + 28j$, $0 \leq j < n$	$0, 9, 15, 26, 27$
second B	$102 + 28n$	F_2
second A run	$204 + 28n + 28j$, $0 \leq j < m$	$0, 9, 15, 26, 27$

$$F_0 = \{2, 3, 4, 9, 20, 21, 23, 24, 26, 28, 29, 37, 44, 45, 46,$$
$$55, 61, 72, 73, 74, 83, 89, 100, 101\}, \qquad F_2 = F_0 \setminus \{9\}.$$

The supplement's `verification/defect_resource_spectrum/code/check.py`, function `check_parametric`, proves equality of these four rows with the complete first-left time set on the full feasible region. All supplement paths here and below are relative to the immutable defect project identified in the Reproducibility section.

5.2 Three independent fold and consumer parameters

Use the inherited 28-letter word A, 102-letter connector B, and

$$P = \mathrm{RLRRRRLRRRRLLLLRLRRL}.$$

From $(-1, 0)$ facing west consider

$$w(a, b, r) = B[9:]A^a B A^b P^r, \qquad a, b, r \geq 0.$$

Here $B[9:]$ deletes nine initial letters. The input field is synthesized from first reads, so the three exponents are independently quantified.

Theorem 5.3 (Exact fold–consumer spectrum)**.** *The word is feasible exactly when*

$$a \geq 1, \qquad 0 \leq b < a, \qquad 0 \leq r \leq b + 2.$$

On this domain, its minimum initial-black cost and contained-defect count are

$$C = 46 + 5a + 5b, \qquad D = \begin{cases} 0 & r = 0, \\ 9r - 3 & r \geq 1. \end{cases}$$

Its resource image is exactly

$$(51, 0) + \mathbb{N}(5, 0),$$
$$(51, 6) + \mathbb{N}(5, 0),$$
$$(51, 15) + \mathbb{N}(10, 9) + \mathbb{N}(5, 0).$$

Consequently $10D \leq 9C - 309$*, with sharp slope and intercept. For a complete input of mass* $m \geq 51$*, the largest possible count within this family is*

$$D_{\max}(m) = 15 + 9 \left\lfloor \frac{m - 51}{10} \right\rfloor.$$

Proof. Compile every phase of the five blocks into its exact position–time relation. Same-cell successor alternation gives a Presburger feasible set; the supplement checks its equality with the displayed region, with no bound on a, b, r. First-left times are partitioned into disjoint residue intervals. Each proposed interval is accepted only after exact set equality with the corresponding fibre, so its integer length counts all first-left reads. Their sum is $46 + 5a + 5b$. The shifted ten-letter matcher includes all junction windows and zero powers, and its time-set count gives the stated D. These are exact symbolic identities in the supplied finite certificate. The declarative file `verification/schema_supply/witness.json` uses schema `flat-schema-supply-v1`; its SHA256 is

`5792fad3124ba6293edfdce5fedc1850e42d5304927edd7f56e4c8853b895b0a`.

The checker `verification/schema_supply/check.py` constructs the `archive` schema in `schemas()` and, in `check()`, compares the computed feasible, supply and defect relations with those three declarations. It checks equality of the projected joint relation with the linear components, the mass optimum and equality ray. The compiler `verification/schema_supply/supply.py` derives same-cell successors and first-left time fibres; its interval/residue counting routine accepts a piece only after exact set equality. These universal arithmetic checks trust the Python compiler and native ISL/GMP. The separate `independent.cpp` implementation checks finite literal trajectories as an independent control.

For $r = 0, 1$, put $b = 0, a = 1 + s$ to obtain the first two components. For $r \geq 2$, put $r = q + 2$; its least feasible b, a are $q, q + 1$. Increasing a supplies all extra multiples of five, yielding precisely the third component. The inequality holds on all three components and is equality at $a = b + 1, r = b + 2$ for every $b \geq 0$, proving both sharpness claims. For the mass optimum take $b = \lfloor (m - 51)/10 \rfloor$, $a = b + 1, r = b + 2$; place any remaining black cells outside the finite read footprint. □

This extends the symmetric producer family used in the factor paper to all independently sized folds and finite consumer trains. Its terminal black mass is $67 + 9(a + b) + 4r$, so initial supply and final mass measure different resources.

6 Small positive witnesses and unary NP membership

Semilinearity provides a decision procedure. The fixed geometry of the cycle library also gives a quantitative size bound on a positive certificate. Let $E(m, N)$ mean that some complete seed of at most m black cells produces at least N contained defects, with all coordinates unrestricted. Put

$$s = 5(N+1), \quad K = 2570(N+1), \quad d = s+2m, \quad H = 2^{24}(K+d+1), \quad R = 64((d+1)!H^d+1).$$

Theorem 6.1 (Small witnesses)**.** *If $E(m, N)$ holds, it has a witness whose initial coordinates and schema exponents have absolute value at most R. Its Nth defect occurs within $K(R + 1)$ updates. In particular, $\log R = O((m + N + 1) \log(m + N + 2))$. With m, N in unary, $E(m, N)$ belongs to NP.*

The radius is existential: it produces a possibly different input with the same budgets. A specified input may put its next obstacle arbitrarily far away.

6.1 A polynomial verifier for a compressed physical run

The canonical cover has at most s powers and $2048(N + 1) + N$ literal letters. There are at most K phase types: the literal bound plus at most $104 \cdot 5(N + 1)$ powered phases is $2568(N + 1) + N \leq 2570(N + 1)$. A powered type has position $A_i(k) + jD_i$ and affine time, with $0 \leq j < k_i$. The nonzero strides, after rotations, are

$$(\pm 2, \pm 2), \quad (\pm 2, 0), \quad (0, \pm 2), \quad (\pm 4, 0), \quad (0, \pm 4).$$

For a proposed phase occurrence, its read colour is initial membership XOR the parity of every earlier type visiting the same cell. Each nonzero-stride type visits a given cell at most once. Solve $A + jD = B + \ell E$ for each pair of types. Nonparallel strides give at most one integral pair, with determinant dividing 16. Parallel strides yield a bounded interval in one residue class of j, of modulus at most two. A source point or literal type gives at most one hit.

Different blocks have their fixed chronological order. Within one powered block, intersecting phases have $\ell = j + c$ for a constant c; the sign of $|u|c +$ old phase $-$ current phase decides precedence. Thus all previous flips are included. Split the current index by its hit moduli, sort

interval starts and one-past-ends, and check the parity on each nonempty interval against the requested bit. Induction on ordinary update time proves that these checks are necessary and sufficient for execution on the complete supplied field.

There are $O(K(K+m))$ pair and point queries and $O(K(K+m)\log(K+m)+m^2)$ comparisons, including source distinctness. All operate on binary integers of polynomial length. A word summary stores length, first and last nine symbols, and contained-defect count; concatenation checks only crossing windows and binary powering is logarithmic. In this library all powered words are cyclically ten-legal and have length at least 24, so the count is fixed once every exponent is classified as zero or positive.

6.2 Linear chambers through successful certificates

Let θ list the independent exponents and the coordinates of $m' \le m$ distinct initial black cells; its dimension is at most d. At one successful witness fix $\theta \bmod 64 = \rho$ and write $\theta = 64z + \rho$. Follow the verifier and record every comparison with its witnessed outcome, including endpoint ordering, equalities, source distinctness, and exponent zero/positive cases. Strict integer inequalities become non-strict inequalities by a unit shift. For a failed equality record its strict sign.

All modular tests are now constant. Determinant denominators divide 16, point-membership denominators divide four, and the parallel-index and parity splits each divide by at most two; the accumulated divisors divide 32. Floors and ceilings are affine in z, with only the constant term rounded. Every integer solution to the resulting chamber follows the same comparison path, hence preserves physical feasibility and exact defect count.

For completeness, the coefficient envelope can be followed explicitly. Set $H_0 = 2^8(K + d + 1)$. After the substitution, origin variable coefficients are at most 256 and their constants at most $K + 252s$. Origin differences fit $2H_0$. Determinant numerators fit $16H_0$ and their quotients fit $16H_0 + 1$. Parallel-hit endpoints fit $4H_0 + 16$; after parity splitting and successors all endpoints fit $16H_0 + 32$. Minima and maxima select one existing form. Endpoint comparisons and collinearity guards fit $128H_0 + 128$. Within-body origin differences cancel all parameters. Thus every integer coefficient and right-hand side is bounded by $256H_0 \le H$.

Proof of Theorem 6.1. The integer small-solution theorem gives a solution to every nonempty n-variable chamber of coefficient height $a \ge 1$ with coordinates at most $(n+1)!a^n$ in absolute value [6]. Apply it with $n \le d, a \le H$, then substitute $\theta = 64z + \rho$. This proves the stated R. The source still has distinct cells, the powers remain nonnegative, and its word still ends at the Nth defect. Summing at most K phase types gives the duration bound. Taking logarithms yields the bit bound.

For unary budgets guess the schema, source coordinates and exponents inside these bounds. They use $O((m + N + 1)^2 \log(m + N + 2))$ bits. Check the conservative schema-size limits, library membership, exact physical execution and defect count using the preceding algorithm. Every accepted certificate is a genuine run, and the finite cover plus small-solution argument supplies one for every positive instance. This proves NP membership. □

The chamber argument applies to the canonical library with free initial coordinates. A supplied affine input family with larger coefficients requires its own coefficient-height bound.

Corollary 6.2. *The minimum initial mass $\mu(N)$ needed for at least N contained defects can be computed in deterministic time $2^{O(N^2 \log(N+2))}$.*

Proof. The word $(\mathrm{LRRL})^q$ has minimum supply $2q$ and $\max(0, 4q - 9)$ defects, so $\mu(N) \le 2\lceil (N + 9)/4 \rceil = O(N)$. Enumerate the bounded positive certificates for successive masses through this upper bound. The preceding bit-size estimate gives the claimed time. □

Short schemas and small integer solutions are established tools for flat counter systems [7]. Here the mutable-field interval verifier supplies the additional ant-specific execution argument.

7 A polynomial-size quantifier-free decision backend

The general Presburger construction supports arbitrary finite-fibre source predicates and complete target equality. For the modal schemas in the finite-defect cover, a narrower syntax admits a polynomial-size existential encoding. This provides an executable decision backend and a second route to the small-witness conclusion.

Quantifier-free encodings of counting properties on flat runs already appear in Decker–Pirogov [3]. Here the additional obligation is to encode all spatial coincidences and chronologically earlier flips in a complete mutable field. The restricted strides below make those parity constraints polynomial in the supplied schema size.

Let

$$\mathcal{D} = \{(\pm 2, \pm 2), (\pm 2, 0), (0, \pm 2), (\pm 4, 0), (0, \pm 4)\}.$$

Every powered word must restore heading and have full drift in $\mathcal{D}$. The source is a finite XOR of segments $a + jv$, $0 \le j < L$, with $v \in \mathcal{D}$, and singletons. Anchors, lengths and exponents may be affine expressions in shared integer parameters; lengths and exponents are constrained nonnegative. Explicit m-point inputs additionally require pairwise distinct coordinates. No coordinate or repetition bound is imposed.

Theorem 7.1 (Quantifier-free physical realization)**.** *For this syntax, a polynomial-time construction produces a quantifier-free linear-integer/Boolean circuit whose existential satisfiability is equivalent to literal execution of the complete prescribed word. Exact contained-defect counts, the final defect flag, endpoint time and pose, and any polynomial number of endpoint-colour queries have polynomial-size circuits as well.*

Proof. Represent each phase of a power by the injective read stream $a_i + jd$, $0 \le j < k$, with its fixed required colour. Its actual colour is the XOR of the source and every earlier planned write at that point. For streams in different blocks, temporal order is fixed. Within one power of period P, a phase-r occurrence at index $j+s$ precedes phase i at index j exactly when $Ps+r < i$. This retains the intra-loop chronology rather than grouping all occurrences by phase.

For a source or write stream $b + \ell v$, solve $a + jd = b + \ell v$ with its finite index bounds. Nonparallel strides give at most one hit by the determinant formula. Parallel strides give a guarded arithmetic interval in j with modulus $|v_\alpha| / \gcd(|d_\alpha|, |v_\alpha|)$, which is always 1 or 2 for $\mathcal{D}$. Splitting $j = 2z + e$ leaves a guarded ordinary interval. Floors and exact divisibility use only fixed integer divisors and therefore have constant-size linear defining constraints.

On a target residue interval $[0, K]$, the XOR of these interval contributions is a step function. It has the required constant colour exactly when its value at zero is correct and every active interior boundary has even total jump parity. Check this at each candidate interval start and one-past-end, including all coincident boundaries. For q contributions, the equality and parity tests use $O(q^2)$ shared wire operations; no ordering of symbolic endpoints is enumerated. With R read streams and M source terms the parity stage has size $O(R(R+M)^2)$, apart from shared affine expressions. Induction through the finite planned word proves that these colour tests are sufficient as well as necessary for physical execution.

For lexical counts, retain the preceding at most nine turns, their length, the defect count and the final-event flag as shared registers. For a power u^k, at most $\lceil 9/|u| \rceil$ copies precede suffix stabilization; later copies contribute a fixed cyclic defect count. Constant-size guarded transfers select the short cases and the stabilized case. The $k = 0$ branch preserves the entire incoming state. Sharing these transfers between blocks gives a circuit linear in spelled length and block count, instead of a Cartesian list of zero/positive choices.

Finally, an endpoint-colour query XORs initial membership with membership in every read stream. This retains all unqueried remote cells. Named gates and linear definitions preserve polynomial size throughout and prove the theorem. □

Corollary 7.2 (Unary budgets with binary side conditions)**.** *For explicit m coordinate pairs, unary $N \geq 1$ and a quantifier-free linear-integer side condition Q of binary length b, existence of a complete m-black source whose Nth contained defect satisfies Q is in NP. Positive witnesses have polynomial bit length in $m + N + b$. For unary N, the minimum initial cost $\mu(N)$ is in* $\mathrm{FP}^{\mathrm{NP}}$.

Proof. The inherited complete cover supplies a schema with at most $5(N+1)$ powers and $2048(N+1)+N$ fixed letters. Its powered bodies are phases of W, T, A, with strides in $\mathcal{D}$. Guess this polynomial-length schema, apply the theorem to the m distinct source points, and require exactly N defects with a bad last letter and condition Q. Guessing Boolean branches reduces the circuit to polynomially many linear integer constraints. The integer small-solution bound used in Section 6 supplies polynomial-bit witnesses, which can be checked in polynomial time. This uses the complete physical cover; satisfiability of a graph relaxation alone would not suffice.

The finite row construction $(LRRL)^L$ gives $\mu(N) \leq 2\lceil (N+9)/4 \rceil$. A finite prefix can be padded by unread black cells outside its footprint, so the NP query for exactly m points decides $\mu(N) \leq m$ without additional side conditions. Binary search over this $O(N)$ range needs $O(\log(N+1))$ NP-oracle queries of polynomial unary-N size. This is a complexity classification; no numerical table of μ is computed here. □

The encoding applies to the stated segment-source syntax and endpoint colour queries. The general compiler handles quantified source predicates, zero-displacement powers and complete-field target equality. Polynomial circuit size does not imply a polynomial deterministic solver, and the NP statement is not a claim for binary-encoded N.

The reproduction archive checks the encoding by exact intersection identities and complete finite endpoint comparisons. Its Z3 and ISL/GMP backends are part of the computational evidence described in the reproduction section.

8 Finite targets and coordinate dependence

The minimum initial mass introduced in Section 1 can equivalently be computed from finite feasible words:

$$\mu(N) = \min\{c(w) : w \text{ is finite feasible and } d(w) \geq N\}.$$

Indeed every input supplies the first-left reads of each prefix, and every feasible word has the minimal seed in (4). A row scanner $(\mathrm{LRRL})^q$ uses $2q$ initial black cells and has $\max(0, 4q-9)$ defects. Hence

$$0 \leq \mu(N) \leq 2\left\lceil \frac{N+9}{4} \right\rceil.$$

Theorem 1.1 makes every fixed-mass query decidable, so this finite upper bound proves that μ is a total computable nondecreasing function. Corollary 6.2 supplies a quantitative computation bound for this function.

The finite-support Highway Conjecture asks for $\bigcap_N E_{m,N} = \varnothing$ for every m, whereas a coordinate-independent event budget asks that some individual $E_{m,N}$ be empty. Nested semilinear sets $\{a : a \geq N\}$ show that these quantifiers cannot be interchanged. A family of increasingly remote finite witnesses need not contain one infinite counterexample.

The synthesis theorem fixes the requested event count and quantifies over all coordinates. The supply dichotomy instead fixes a finite schema collection and allows unbounded repetitions. Keeping these two inputs explicit gives a way to study which physical mechanisms can produce many defects at low cost.

9 Limitations

The all-coordinate construction takes fixed numerical budgets m, N; its formulas need not have uniformly bounded size when these budgets vary. The supply dichotomy applies to a supplied finite schema collection. Its sharp constants describe that collection, and a growing family of finite witnesses is distinct from a single infinite trajectory. Fixed-prefix decision procedures alone do not settle eventual convergence.

The polynomial-size quantifier-free backend assumes the heading-restoring stride library and source syntax specified in Section 7; the complexity statements use unary budgets. Unary time-fibre counting is essential: general planar Presburger fibres can have nonlinear cardinality. The proof gives an effective universal disjunction for nonzero budgets; the reported computations verify the finite cover and supplied schemas rather than expand every instance of that disjunction. Exact programs and written reductions constitute the computational proof, with language and arithmetic-library implementations in its trust base.

10 Reproducibility

Run `python3 defect_synthesis/reproduce.py`. The self-contained supplement contains two exact relation compilers, independent finite-state controls, the finite-cover reconstruction, the complete three-parameter resource identity and fixed-schema query optimizations. Python 3.10+, a C++17 compiler and the system ISL/GMP and Z3 shared libraries are required. The arithmetic backend performs exact integer set operations [4]; the retained computational reports use ISL 0.27.

The computational supplement is fixed at repository commit `e932522`, defect project. Its `sources.json` lists the exact source and certificate hashes. The imported factor results are Theorems A–B (attraction), Subsection 3.3 (the all-parameter fold obstruction), and Theorem F with its finite covering lift in Subsections 4.1–4.2 of the archived article [5].

The programs check supplied-schema identities and reconstruct the finite cover. The universal nonzero-budget disjunction is established by the constructive proof; its full expansion is not needed by these computations. The small-witness theorem follows from the interval and chamber argument in Section 6.

For repeated integer threshold queries, the optimized kernel represents a unary time set by its strict-order relation. Its mth relation power witnesses $m+1$ distinct times, so binary powering reduces a fresh threshold query to $\lfloor \log_2 m \rfloor + \mathrm{popcount}(m) - 1$ compositions for $m \geq 1$. A shared decision graph also evaluates the 84 allowed ten-factors with 183 mask intersections instead of 840. Exact set-equality controls compare the optimized queries with their baseline semantics. These improvements preserve the mathematical schema cover and the complete endpoint field.

A Fixed turn words

The following representatives use the origin and north-facing initial heading, with N,E,S,W step vectors $(0,1), (1,0), (0,-1), (-1,0)$. Line breaks carry no meaning. A cyclic phase is a rotation of the letters; its displacement is a quarter-turn rotation of the displayed representative's vector when the initial heading is reset to north.

W (104 letters):

```
RRLLLLRLRRRRLRRRRLLLLRLRRRRLRLLRRLLLLRRLLRRRRLLRRLRL
LRLLRLRLLRRRRLLRLLRRRRLLRRRRLLRLRRRRLRLLLLRRRRLRRLRR
```

T (24 letters):

```
RRRRLLRLLRRRRLLRRRRLLRLL
```

A (28 letters):

LLRRRRLRRLRRRRLLLLRLRRRRLRLL

B (102 letters):

RRLLLLRRLLRRRRLLRRLRLLRLLRLRLLRRRRLLRLRRRRLRLLLLRRRR
LRRLRRRRLLLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLL

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