Local Factor Rigidity and Mass–Defect Bounds for Langton's Ant

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Abstract

Can the short turn patterns of a highway force a Langton ant to follow that highway? We call a ten-turn window legal when it occurs in the standard 104-turn highway word, read cyclically. If every future window is legal and the initial colouring has m black cells, the ant reaches the standard highway within $12m+212$ updates. The coefficient 12 is optimal. For trajectories from any finite black set, the same language measures departures from highway behaviour. Writing B_t for the black cells at time t and $N_{10}(s,t)$ for the illegal windows wholly contained in $[s,t)$, we prove

$$3(t-s)-26(|B_t|-|B_s|)\le 351N_{10}(s,t)+322.$$

Thus a trajectory with few illegal windows must create black cells at nearly the highway rate. The additive constant 322 and the asymptotic mass-growth slope are sharp. The attraction proof excludes arbitrarily repeated folds in a finite graph of physically feasible words. The mass inequality uses a separate finite-word potential and applies even to fragments with no infinite legal continuation. We also compute the next illegal window and its complete output state, describe the geometry of finite legal intervals, and show that nine is the least window length giving a uniform legal-word mass-deficit bound.

1 Introduction

The standard Langton highway repeats a sequence of 104 left and right turns. A much shorter observation records only which ten-turn windows occur in that cycle. Does matching this local vocabulary force the ant eventually to follow the whole highway? The question involves the physical colouring as well as the turn sequence: each visit flips a cell, so a word allowed by overlapping windows may still request the wrong colour when it returns to an earlier site.

We answer this question for every finite initial black set. If all future ten-turn windows belong to the highway vocabulary, the ant enters the standard highway within $12m+212$ updates, where m is the initial number of black cells. The linear coefficient is optimal. On an infinite initial colouring, two other eventual modes can occur. The proof identifies these possibilities in a finite word graph and then excludes arbitrarily long combinations of folds by explicit conflicts at repeated cell visits.

The same vocabulary yields a second result for trajectories with no promise about their future turns. Every illegal ten-turn window contributes a fixed allowance to a

bound on the difference between elapsed time and black-cell growth. Consequently a long interval with few violations must leave nearly as many black cells as a highway would. This statement requires a separate proof for finite words: a finite legal fragment need not admit a legal infinite continuation. A finite graph potential supplies that proof.

After establishing these two results, we develop their algorithmic and geometric consequences. We compute the first local violation and the complete state it leaves, bound the geometry and black supply of finite legal intervals, and construct examples that store and later read back black memory. Nine-letter windows suffice for a uniform mass-deficit bound, whereas ten are necessary to force eventual periodicity on arbitrary colourings. These are different thresholds for different conclusions.

Gajardo's symbolic projection and forbidden-word framework [1] provide the finite-word viewpoint. Gajardo and Mazoyer [4] relate finite-state trace descriptions to restrictions on a head's return tours; their one-dimensional equivalences are in Section 6 of the cited author text. Gajardo, Lutfalla and Rao [2] study highway multiplicity for generalized ant rules; Jillhewar's full-drift chronology [3] supplies the periodic realization test used here. Our contribution is the concrete physical classification of the short-factor language, its sharp quantitative bounds, and the resulting local-event algorithms. The all-parameter exclusion of repeated folds and the finite-word potential are the two main proof steps.

The paper first proves attraction and the mass inequalities. It then develops exact event computation and finite-interval geometry. The appendices give the explicit words, supply tables, and memory constructions used to attain or illustrate the bounds.

2 Model and principal results

A state consists of the complete black set B, a position in $\mathbb{Z}^2$, and a cardinal heading. White turns right (R), black turns left (L); the cell then flips and the ant moves. Time denotes the state before the update. Denote the state by $X = (B, p, h)$ and one update by F. Headings N,E,S,W correspond to $h = 0, 1, 2, 3$ and vectors $(0, 1), (1, 0), (0, -1), (-1, 0)$. The update is invertible: step back along the current heading, flip that cell, and undo the turn determined by its recovered colour. Thus each complete state determines its turn sequence at all integer times, including the nine turns preceding the input checkpoint. Let W be the standard 104-letter turn word in the appendix and let $\mathcal{L}_k$ contain all its cyclic factors of length k. A *factor* is a consecutive subword; cyclic factors may cross the end of W and continue at its beginning. A stream is k-legal if every future length-k window belongs to $\mathcal{L}_k$. A physical stream is the turn sequence of an actual ant on a specified colouring. The *standard trace* repeats W, together with its induced positions, headings and read colours, up to translation, rotation and cyclic phase. The entry time $\tau(X)$ is the earliest update from which this complete future trace is standard. The mass is $|B_t|$; unvisited initial black cells remain part of that set. Write $H(X)$ for $\tau(X) < \infty$. The vocabulary $\mathcal{L}_{10}$ contains 84 distinct words. Forward legality tests windows starting at times $t \geq 0$; the intrinsic predicate J_{10} tests the window ending now,

including inverse-recovered turns near the input checkpoint. The contained count $N_{10}(s,t)$ tests only windows wholly inside the specified interval. These conventions agree away from its first nine updates. For finite words, we also use *silent* to mean ten-legal; an event-free trajectory interval excludes the intrinsic J_{10} tests at the times specified.

Theorem A (eventual modes). Every physical ten-legal stream on an arbitrary initial colouring eventually has least period 104, 28 or 24, with the respective words W, U, T given below.

Theorem B (finite-mass attraction). For a complete finite initial black set of cardinality m, a ten-legal future reaches the standard trace by

$$\tau(X) \le 12m + 212.$$

The coefficient 12 is optimal: a family with $m_n = 2n + 13$ has $\tau_n = 24n - 3$ for every $n \ge 1$. The intercept 212 is a proved upper bound.

Theorem C (arbitrary-colouring threshold). An explicit computable infinite colouring has a nine-legal stream that is not eventually periodic. Thus ten is the least factor length forcing eventual periodicity on arbitrary colourings. This sharpness statement uses infinite support.

Theorem D (local event). Let J_{10} test the true preceding nine turns and the current turn for membership outside $\mathcal{L}_{10}$, recovering the past by inverse dynamics. On every finite input, existence of a future J_{10} is decidable in polynomial bit time for sparse binary coordinates.

Theorem E (mass–defect balance). On every interval $[s,t)$ of every finite-support trajectory,

$$3(t-s) - 26\big(|B_t| - |B_s|\big) \le 351 N_{10}(s,t) + 322,$$

where N_{10} counts forbidden ten-letter windows wholly contained in the interval. The constant 322 is attained by a 90-step witness; the mass slope $3/26$ is asymptotically sharp. The penalty 351 is not asserted optimal.

Theorems A–C concern infinite legal futures; Theorem D uses the finite-mass attraction bound and a complete-field periodic-tail test. Theorem E instead rests on a complete finite-word potential certificate. This separates their logical dependencies while using the same local language.

3 From local language to eventual attraction

3.1 Finite-word feasibility and graph coverage

For any finite prescribed turn word, the geometric path is fixed after choosing an initial pose. It is physically realizable if and only if consecutive reads of each same physical cell have alternating letters. Necessity is the flip rule. For sufficiency, make a visited cell initially black exactly when its first requested letter is L; leave every unvisited cell white. Induction over the finite word proves execution. Gajardo [1], Section 3, pp. 60–62, gives this first-read assignment and its forbidden-word formulation.

Let E contain every physically feasible 150-letter word whose 10-factors lie in $\mathcal{L}_{10}$. Construct a graph G with edge $v_0 \cdots v_{148} \to v_1 \cdots v_{149}$ for each $v \in E$. No word is removed because it looks unlikely or because its seed is large. Complete enumeration follows all whitelist-compatible extensions and rejects exactly the same-cell alternation conflicts. Induction on word length proves completeness of that enumeration.

Any actual infinite 10-legal stream gives an infinite path in G, since each of its length-150 windows is feasible under its own actual execution. The graph can admit paths that fail longer physical consistency. It is an OVERAPPROXIMATION; this does not invalidate exclusion, but prevents interpreting every graph path as a physical orbit.

The immutable certificate `verification/ten/certificates/windows10.txt` contains 6,605 edge words. Both verifiers regenerate the entire set and require exact equality, including cardinality and absence of duplicates. Their complete graph has 6,501 vertices and precisely three cyclic strongly connected components:

Component	Vertices	Internal edges	Structure
$\mathcal{W}$	104	104	exact standard simple cycle
$\mathcal{T}$	24	24	exact T simple cycle
$\mathcal{H}$	366	370	not a simple cycle

There are no other cyclic components, including self-loops. The first two component edge sets equal all cyclic length-150 factors of W and T. Strongly connected components and reachability are checked on the full graph, not only on a pruned experimental sample.

The non-simple component is crucial: ordinary finite-state periodicity reasoning cannot dispose of it. It is dealt with next by an all-length physical conflict, rather than by assuming its paths are periodic.

3.2 Exact two-loop description of the non-simple component

The certificate `quotient.tsv` assigns each of the 366 vertices of $\mathcal{H}$ a state in $\{0, \dots, 101\}$. It is a label-preserving homomorphism onto this explicit graph:

- for each $0 \le i < 102$, the edge labelled B_i leads from i to $i+1$ modulo 102;
- one additional edge, labelled L, leads from 0 to 75.

Here B is the 102-letter word in the appendix, $B_0 = \mathrm{R}$, and the alternative short loop is $A = \mathrm{L}B_{75} \cdots B_{101}$, of length 28. Every one of the 370 internal edges is checked against this formula. The homomorphism is used only in the forward direction: every infinite internal path projects to a word that, after at most 101 letters, is an arbitrary concatenation of A and B.

The edge label is the letter APPENDED to the 149-letter history. It is shifted 149 places from the original stream. This finite shift is harmless for eventual-mode classification. Subsection 3.4 instead uses the CURRENT letter v_0 of a future-window vertex, and does not confuse these conventions.

The exact geometric data, starting north at the origin, are

$$A : (0, -4),\ h = 0; \qquad B : (1, 11),\ h = 2.$$

Thus A translates and B reverses the heading. The A word is a cyclic phase of U.

3.3 Three foldbacks are impossible, at every pair of lengths

Lemma 1. For every pair of integers $n, m \geq 0$, the finite word BA^nBA^mB is physically infeasible, independently of its initial colouring.

In each case below there are two consecutive visits to a cell q both requesting R:

Parameter range	q	Earlier time	Later time
$n = 0$	$(0, 2)$	32	184
$n \geq 1,\ m \geq n$	$(0, 2)$	32	$184 + 56n$
$n \geq 1,\ 0 \leq m < n,\ r = n - m$	$(1, 4r)$	$25 + 28r$	$205 + 28(n + m)$

All times are pre-update indices in the displayed finite word. It suffices to verify their consecutiveness, not that the rest of the candidate word is feasible.

Proof. Let the five block boundaries start at times 0, 102, $102 + 28n$, $204 + 28n$, and $204 + 28(n + m)$. Their respective poses are $(0, 0), N$; $(1, 11), S$; $(1, 11 + 4n), S$; $(0, 4n), N$; and $(0, 4(n - m)), N$.

The following finite coordinate facts are computed directly from the two displayed words and checked independently: A has $-3 \leq x \leq 0$; on its column $x = 0$ the pairs (index, height, letter) are

$$(0, 0, L), (7, -1, R), (8, -2, R), (15, -3, L), (16, -2, L), (27, -5, L).$$

In B, the only visit to $(0, 2)$ is index 32, R. All B visits at $x = 0$ have $y \leq 2$. Visits to $(1, 9)$ are indices 82,R and 90,L; visits to $(1, 4)$ are 45,L and 53,R; visits to $(1, 8)$ are 73,L and 81,R. On $x = 1$ its maximum height is 12.

If $n = 0$, the reflected second B reads $(0, 2)$ at its local $(1, 9)$, index 82. The previous visit is the first B's index 32. This gives the first row.

If $n \geq 1$, the first A run lies at $x \geq 1$, so cannot revisit $(0, 2)$. In the second B, a visit there would require local $(1, 9 + 4n)$, above every $x = 1$ visit. In the second A run, the first matching copy is number $n - 1$ (zero-based), at local index 8, with letter R. All earlier copies miss: on $x = 0$ their required local height is below -5. The visit time is $204 + 28n + 28(n - 1) + 8 = 184 + 56n$. This copy exists when $m \geq n$.

Finally let $r = n - m > 0$. For $r = 1$ or 2, the last earlier visit to $(1, 4r)$ is the first B's index 53 or 81, both R. The first A run lies at heights at least 11 and cannot reach those two cells. For $r \geq 3$, the last visit in the first A run is copy $r - 3$, local index 7, R, at time $102 + 28(r - 3) + 7 = 25 + 28r$. The column-$x = 0$ list for A shows no later visit to that cell in the run; any visit in the first B precedes it. A visit by the second B would require local $(0, 11 + 4m)$, impossible since its column-$x = 0$ maximum is 2.

The second A run lies at $x \leq 0$. The third B, based at $(0, 4r)$, first reaches $(1, 4r)$ at index 1, R. This proves the third row and the lemma. □

An infinite concatenation of A and B with infinitely many B's contains a forbidden three-B subword. Hence it has only finitely many B's and is ultimately A^ω. This argument disposes of arbitrary interleavings and unbounded run lengths, not only tested values of n, m.

An infinite path in a finite graph eventually remains in one cyclic strongly connected component. In $\mathcal{W}$ and $\mathcal{T}$ it is forced periodic by the simple cycles. In $\mathcal{H}$ the preceding homomorphism and Lemma 1 force an A-periodic suffix. This proves Theorem A.

3.4 Finite support and the transient bound

For a heading-resetting periodic word of length P with nonzero drift d, partition its phase positions by differences in $\mathbb{Z}d$. For a class write $q_i = b + a_i d$. Visits to $b + nd$ are eligible when $a_i \leq n$, and their chronological order is decreasing a_i, then increasing i. A cell's required initial colour is the first letter of that ordered eligible list. This is the full-drift chronology method of Jillhewar (2026), Theorem 3.1 and Remark 3.2.

The words U and T pass all per-cell alternation tests but have respectively five and two stabilized classes starting L. Each such class demands infinitely many DISTINCT initially black cells. Neither word can be executed forever from finite support. Any finite prelude preserves finite support, so these modes also cannot occur as eventual tails of a finite-support orbit. Consequently every finite-support infinite 10-legal path eventually enters $\mathcal{W}$.

The complete graph gives 925 vertices that can reach $\mathcal{W}$. They exclude $\mathcal{H}$ entirely and include $\mathcal{T}$. Apart from $\mathcal{W}$ and $\mathcal{T}$, they are acyclic. A path cannot leave a strongly connected component and later return to it. The current letters of a run of L successive vertices in $\mathcal{T}$ form a length-L prefix of a phase of T^ω.

Let $b_j(L)$ count distinct cells whose first read is L in this prefix starting at phase j. If the complete black mass at the run's start is m_T, physical execution requires $b_j(L) \leq m_T$. The count includes only first reads relative to that run; cells written during the run cannot pay for them.

Lemma 2. For all phases j and all integers $L \geq 0$,

$$L - 12b_j(L) \leq 5.$$

Proof. Chronological classes give the first-L read times as a finite set F_j and two arithmetic rays $r_{j,1} + 24\mathbb{N}$, $r_{j,2} + 24\mathbb{N}$. Their exact table is in the appendix. The sets count distinct physical cells, not repeated L reads. Put $D_j = \max(\{0\} \cup \{f + 1 : f \in F_j\} \cup \{r_{j,1}, r_{j,2}\})$. For every $L \geq D_j$, directly from these rays,

$$b_j(L + 24) = b_j(L) + 2.$$

Thus $L - 12b_j(L)$ is 24-periodic for all $L \geq D_j$. Checking the finite interval $0 \leq L < D_j + 24$ covers all lengths. Both implementations derive the class table independently and verify that the maximum across all phases is 5. This is an exact finite reduction with its all-length periodicity proof, not a finite-prefix extrapolation. □

Before its first $\mathcal{T}$ vertex a path uses only the 54 acyclic strict ancestors of $\mathcal{T}$. Give such a vertex v weight

$$c(v) = 1 + 12\delta(v_0), \qquad \delta(R) = 1,\ \delta(L) = -1.$$

Complete acyclic dynamic programming gives maximum total weight 70 for a path ending at $\mathcal{T}$. Therefore if t_{pre} updates precede that run and their net black change is Δ, then $t_{pre} + 12\Delta \leq 70$ and $m_T = m + \Delta$. After departure from $\mathcal{T}$, the maximum acyclic path length before $\mathcal{W}$ is 137. The exit-producing update from a $\mathcal{T}$ vertex is included in its T-prefix, not omitted. By Lemma 2,

$$\tau \leq t_{pre} + (12m_T + 5) + 137 \leq 12m + 70 + 5 + 137 = 12m + 212.$$

Paths avoiding $\mathcal{T}$ altogether have length at most 167 before $\mathcal{W}$, also below this bound. A path whose initial vertex is already in $\mathcal{W}$ has zero such entrance time. Once the path first reaches $\mathcal{W}$, leaving it would preclude a later return; because its eventual component must be $\mathcal{W}$, it never leaves. Its future is standard from that time.

These are future-window vertices at times starting at zero, so the transient bound needs no additional window warm-up. Every complete initial black cell remains in the physical state; only its count is used in this inequality. Theorem B's upper bound follows.

3.5 An explicit family attaining coefficient twelve

Define

$$E = \{(-2,-1),(-1,0)\},$$

$$\begin{aligned} D = \{&(-1,-4),(-1,-3),(0,-5),(0,-3),(0,-2),(1,-6),\\ &(1,-5),(1,-4),(1,-3),(2,-3),(2,-2)\}. \end{aligned}$$

$$R = \{(-2,0),(-1,-1),(-1,0),(0,-1),(1,-1),(1,0)\}.$$

For $n \geq 0$, put

$$B_n = E \ \cup\ \{(-2,-2j),(0,-2j-1) : 1 \leq j \leq n\} \ \cup\ (D + (0,-2n)).$$

These sets are disjoint. Start north at the origin with exactly B_n black and all other cells white. Then $|B_n| = 2n + 13$.

The word T has read footprint within $[-2, 1] \times [-3, 0]$, net displacement $(0, -2)$, and resets heading. For every $n \geq 1$ its required initial black cells in that footprint are precisely $E \cup \{(-2, -2), (0, -3)\}$. All later rail cells and all cap cells are outside the footprint. The complete 24-step parity of visits, followed by set cancellation, gives the identity

$$F^{24}(B_n, 0, N) = \big((B_{n-1} + (0, -2)) \,\dot\cup\, R,\ (0, -2), N\big).$$

This is an identity of FULL states, not a core-only recurrence. A direct all-parameter derivation is: translate the rail terms with $j = 1, \ldots, n - 1$ by -2; they become the original terms $j = 2, \ldots, n$. The cap becomes $D + (0, -2n)$. Only the fixed finite set E, the first rail pair, and the translated E remain to compare, and their difference with R equals the exact net toggle set of T. This finite equality is checked. It also covers $n = 1$ where the shorter rail is empty.

The residual R is at heights -1 and 0. Every future repeated T segment based at height -2 or less remains strictly below it. The $n = 0$ complete state is the clean seed of the standard phase Z in the appendix; its full infinite execution is verified by chronology and an independent epoch-breakpoint formula. Its phase footprint has maximum height zero and drift $(-2, -2)$, so its continuation from height -2 also stays below R. Induction therefore retains and guards all successive residuals and proves the exact infinite word

$$T^n Z^\omega.$$

The factors of $T^2 Z^2$ verify every possible length-10 junction; T^ω and Z^ω are themselves 10-legal. The last three letters of T and Z agree, but the last four do not. Since Z has least period 104, any earlier standard future agreeing with Z^ω must be its unique phase-aligned left extension. Hence the first standard time is

$$\tau_n = \max(0, 24n - 3).$$

For $n \geq 1$, $\tau_n = 12|B_n| - 159$. An affine upper bound with slope less than 12 and fixed intercept fails for large n. This proves the sharp coefficient, without claiming 212 is sharp. Every member of this family eventually reaches the standard highway.

3.6 Nine factors admit an aperiodic infinite-background spiral

Let U and C be the 28- and 89-letter words in the appendix and let $n_i = 5 \cdot 2^i$. Consider

$$a = \prod_{i=0}^{\infty} U^{n_i} C.$$

The prescribed geometry is explicitly computable, starting at the origin heading north. Its all-future physical feasibility follows from the next separation proof; it is not assumed.

A U block restores heading, moves $(0, -4)$, and has footprint in $[0, 3] \times [-5, 0]$. A C block rotates the heading counterclockwise, moves $(13, -4)$, and has footprint in $[0, 14] \times [-4, 0]$. The finite word UCU is feasible. Every length-9 factor of U^2CU^2 is in $\mathcal{L}_9$.

The start heading of macro i is $-i$ modulo four. For $S = 16^k$, the four consecutive start positions are

$$p_{4k} = (-8S + 8, 4S - 4), \quad p_{4k+1} = (-8S + 21, -16S - 8),$$
$$p_{4k+2} = (32S + 25, -16S + 5), \quad p_{4k+3} = (32S + 12, 64S + 9).$$

These follow by the two block displacements; one group of four returns the heading and replaces S by $16S$. Entire macro read footprints lie in the respective rectangles

Index	x interval	y interval
$4k$	$[-8S + 8, -8S + 22]$	$[-16S - 8, 4S - 4]$
$4k + 1$	$[-8S + 21, 32S + 25]$	$[-16S - 8, -16S + 6]$
$4k + 2$	$[32S + 11, 32S + 25]$	$[-16S + 5, 64S + 9]$
$4k + 3$	$[-128S + 8, 32S + 12]$	$[64S - 5, 64S + 9]$

Each rectangle is disjoint from all nonadjacent earlier rectangles. For the first row, $k \geq 1$, earlier rectangles through $4k-2$ have $x \geq -S/2+8$, while the new maximum is $-8S+22$; use $15S > 28$. For the second row, earlier rectangles through $4k-1$ have $y \geq -S - 8$, while the new maximum is $-16S + 6$; use $15S > 14$. For the third row, earlier rectangles through $4k$ have $x \leq 2S + 25$, while the new minimum is $32S + 11$; use $30S > 14$. For the fourth row, earlier rectangles through $4k + 1$ have $y \leq 4S + 9$, while the new minimum is $64S - 5$; use $60S > 14$. The first two empty-earlier-set base cases are automatic. These inequalities hold for every $S = 16^k$ in the stated ranges.

It remains to control adjacent macros, which are not claimed disjoint. In local coordinates of the earlier macro, the next macro begins at $(13, -4n_i - 4)$ heading west. Its first U block can overlap the earlier C, but every later U block has $x \geq 17$ while C has $x \leq 14$; the next C is still farther right. The preceding U rail has $x \leq 3$, so is disjoint from the next macro. Within one macro, C can meet only the LAST U block: every earlier U block has minimum height at least $-4n_i + 3$, above C's maximum $-4n_i$. Thus all interactions with C are contained in the verified word UCU. Other repetitions are portions of the physically feasible U^ω trace. This proves alternating letters at every consecutive same-cell pair in the entire infinite word.

Colour a cell initially black precisely when its first prescribed read has letter L; leave unvisited cells white. The proved consistency makes this a genuine infinite classical-ant execution by induction over time. This colouring is computable: for a queried cell

of infinity-norm M, choose k with $16^k > M + 100$. Every rectangle from macro $4k$ onward lies outside $[-M, M]^2$ in at least one coordinate, so only the first $4k$ finite macros need be searched. It has infinite black support, since interior portions of arbitrarily long U rails demand fresh initial black cells.

The word $D_{bad} = \texttt{RLRRRRLRRL}$ is absent from $\mathcal{L}_{10}$ and occurs exactly once in each connector C, at offset 11. The finite check of U^2CU^2 excludes other occurrences at either junction, and U^ω has none. Successive occurrences are separated by $89 + 28n_{i+1}$, which grows without bound. An eventually periodic sequence with infinitely many occurrences of a fixed finite factor has bounded gaps between them. Hence a is not eventually periodic. This proves Theorem C, including its explicitly infinite-background boundary.

3.7 A radius-nine target decidable without waiting for a remote collision

The inverse of a literal step is unique: from post-pose (p, h) set $q = p - \mathrm{dir}(h)$, undo the flip at q, and undo the turn dictated by that recovered colour. Nine inverse steps read only cells within distance nine of the current head. Therefore the preceding-nine-turn test J_{10} is an intrinsic finite-local predicate, including at input time zero.

Let $M = 12|B_0| + 212$. The following total algorithm acts on an arbitrary finite input, with no legality promise:

1. Recover the true nine-turn past. Simulate up to time M, checking J_{10} before each update. Return YES with its first time if encountered.
2. At time M, test the complete-state predicate P_{now} for permanent standard P104. If it fails, return YES (a future event is forced by Theorem B). This is not an assertion that the event was observed or that its first time has been computed.
3. If P_{now} holds, simulate nine additional updates to check all windows crossing this boundary. Return YES if one is bad; otherwise return NO. All subsequent windows are inside the certified standard future.

Why is the second step sound? Absence of every future J_{10} implies legality of every forward length-10 window from time zero. Theorem B would force P_{now} by time M, contradicting the failed test. A far-away blocker cannot be omitted from that test.

The exact P_{now} test first simulates 104 turns on a copy. If they do not equal any phase of W, it returns false. Otherwise let K be the unique clean finite seed for that placed phase, computed by the chronological class formula. Let $q_i + \mathbb{N}d$ be its 104 future phase rays. The actual complete field realizes that future if and only if

$$(B \triangle K) \cap \bigcup_{i=0}^{103} (q_i + \mathbb{N}d) = \emptyset.$$

Sufficiency follows by first-difference induction; necessity follows at the first read of an initial differing cell. Testing finitely many complete-field points against 104 integer rays is exact, including arbitrarily distant markers. No repeated-prefix heuristic is used.

During the $O(m)$ ordinary updates the complete field has $O(m)$ cells, and coordinate bit lengths grow by only $O(\log(m+1))$ beyond their input sizes. Exact finite-set operations and integer ray tests therefore give a deterministic polynomial bit-time implementation (a balanced search tree can be used instead of hashing). The implementation uses Python integers and finite sets; even worst-case linear set lookups remain polynomial. There is no claim of a polynomial bound for the numerical first-event time or of a bound polynomial in the bit length of a COMPRESSED black count.

3.7.1 A fixed-mass late-event control

`verification/ten/certificates/far_marker.json` gives a complete 27-black standard state at a mature phase. Add one black marker at $(2n, 2n)$ for any $n \geq 20$. The new complete mass is always 28. The first contact is exactly at update $104n - 92$, and it changes a prescribed R into L, forming the forbidden ten-factor `RRRLLLLRLL`.

The all-parameter first-contact calculation is: for the supplied phase positions q_i, write $q_i = s_i(2, 2)$ whenever possible. The first visit to the marker has time $104n + \min_i(i - 104s_i)$. The minimum is -92, attained at phase 12, and all relevant epochs are nonnegative for $n \geq 20$. The marker is outside the finite true nine-step inverse history and the seed. Until contact, first-difference coupling preserves the entire standard path. The standard cyclic preceding nine letters and the changed L give exactly the stated bad factor.

For $n = 10^{12}$ the first event is at 103999999999908, but the new decision algorithm returns a forced-future YES after only 548 main-trajectory updates. The extra operations are nine inverse updates and the constant lookahead and ray tests, not simulation to the event. This family illustrates the distinction between reachability decision and event-time observation. Its coordinates encode the long waiting time.

3.8 Finitely many violations and standard convergence

For finite inputs, a standard future has no intrinsic violations after its first nine updates. Conversely, after the last violation every forward window is legal; Theorem B applied to that complete state gives a standard future. Hence

$$H(X) \iff \#\{t \geq 0 : J_{10}(F^t X)\} < \infty.$$

Theorem D decides each future-event query. This equivalence identifies the additional global assertion needed for convergence: finiteness of the event chain, rather than decidability of an individual query.

4 Mass and defects on arbitrary intervals

For a finite word v, put $g(v) = \#R - \#L$ and $D(v) = 3|v| - 26g(v)$. **Theorem (finite-word deficit).** Every physically feasible, ten-legal finite word satisfies $D(v) \leq 322$, with no infinite-extension premise. We derive the all-input inequality of Theorem E from this finite-word bound.

4.1 Exact finite-word feasibility and enumeration coverage

A prescribed turn word fixes its geometric path from $(0, 0)$, heading N. It is physically feasible exactly when successive reads of each physical cell request opposite turns. Necessity is the flip rule. For sufficiency, put an initial black cell precisely at each visited position whose first request is L, and leave all other positions white. Induction realizes the entire word. The required seed is finite. Arbitrary poses follow by translation and quarter turns.

This elementary criterion is prior methodology, as in Gajardo's symbolic forbidden-word treatment. It implies in particular that finite-word feasibility is the same whether initial colourings are required to be finite or are arbitrary.

Starting at the empty word, branch on BOTH turns. Reject an extension exactly if its new ten-factor is forbidden, or its current cell was previously read with the same letter. Induction on length proves complete enumeration of physically feasible ten-legal words through depth 300. Words of lengths below ten are initially unrestricted by the standard factor table.

The Python implementation uses breadth-first cloned last-read maps. The C++ implementation instead uses depth-first reversible current-colour assignments. No seed-size or spatial cutoff is used. Their complete sorted depth-300 edge sets and their integer potentials have identical hashes.

4.2 The finite graph certificate covers arbitrarily long finite words

Form a graph from every accepted word of length 300: its 299-prefix is the source, its 299-suffix the target, and the appended letter labels the edge. There are exactly 29,334 vertices and 29,559 edges. Assign the weight

$$w(R) = -23, \qquad w(L) = 29,$$

so that $D(v)$ is the sum of its letter weights.

The certificate assigns a nonnegative integer $h(u)$ to each vertex, with maximum 251, and verifies every edge inequality

$$h(u) \geq w(c) + h(v) \quad \text{for every edge } u \xrightarrow{c} v \tag{1}$$

and every initial-vertex inequality

$$D(u) + h(u) \leq 284. \tag{2}$$

These are direct integer comparisons; an optimizer's output is not trusted. The potential generators are only proposal generators. Their resource caps raise an error with no mathematical conclusion. An independently supplied damaged potential is rejected without repair.

For any physically feasible ten-legal finite word z of length $n \geq 300$, all its 300-windows belong to this graph. They define a path starting with its first 299 letters u_0. Summing (1) over the appended letters gives

$$D(z) \leq D(u_0) + h(u_0) - h(u_{\text{final}}) \leq 284,$$

using (2) and $h \geq 0$. This is an overapproximation argument: not every graph path need be physically realizable, but every physical word is covered. No extension to an infinite ten-legal path is assumed. There is no misuse of eventual-cycle classification for a finite segment.

For all lengths 0 through 299, exhaustive enumeration gives maximum 322. Combining the two finite kernels proves the finite-word deficit theorem for all finite lengths.

The source witness fixes SHA256 hashes for the sorted depth-300 words and the potentials in lexicographic vertex order.

The complete finite sets are reconstructed deterministically and checked against those hashes and the local inequalities. They need not be stored as a nine-megabyte enumeration dump. `verification/mass/witness.json` contains the immutable numerical claim, exact short witness and its complete minimal seed.

4.3 Sharpness

The word

```
LLLLRLRRRRLRLLRRLLLLRRLLRRRRLLRRLRLLRLLRLRLLRRRRLLRLRRRRLRLLRRLLLLRRLLRRRRLLRRLRL
↪ LRLLRLRLL
```

has length 90, 44 right turns and 46 left turns. It is physically feasible and all of its 81 ten-factors are allowed. Its unique minimal first-read seed has 28 black cells and is listed in `verification/mass/witness.json`. Literal execution leaves 26 black cells. Thus

$$3 \cdot 90 - 26(26 - 28) = 322,$$

proving the additive constant in the finite-word deficit theorem and in Theorem E, with wholly contained windows, cannot be decreased.

The standard word has 104 updates and 12 more R than L. Its certified finite-support realization supplies physically feasible words W^k for every $k \geq 1$. Thus any uniform lower estimate $g \geq \rho n - C$ with $\rho > 12/104 = 3/26$ fails as k tends to infinity. The standard tail is an existing certified input here; this work uses that standard growth as its sharpness witness. The supplementary literal W^4 check is a control, not the justification of the universal k quantifier.

4.4 Derivation of the unrestricted inequality

In any actual finite orbit segment, every right turn increases complete black mass by one and every left turn decreases it by one. Hence

$$g(a_s \cdots a_{t-1}) = |B_t| - |B_s|. \tag{3}$$

Remove each letter that ends a forbidden ten-window wholly contained in the segment. These $N = N_{10}(s,t)$ single-letter positions separate at most $N+1$ consecutive remaining runs. Every ten-window wholly inside a remaining run was wholly inside the original segment and cannot be forbidden: otherwise its last position would have been removed. Each run is a genuine physical subword and is therefore covered by the finite-word deficit theorem. An empty run has deficit zero and may harmlessly be charged the same upper bound.

Each removed letter has deficit at most 29. Additivity of D, the finite-word deficit theorem and (3) yield

$$D(\text{total}) \le 322(N+1) + 29N = 351N + 322,$$

proving Theorem E. This does not require disjoint forbidden windows. Removing their ending letters handles overlaps exactly.

4.5 Density consequences

Taking $s = 0$ and dividing by t gives two immediate consequences: if $|B_t| = o(t)$, then $\liminf N_{10}(0,t)/t \ge 1/117$; if $N_{10}(0,t) = o(t)$, then $\liminf |B_t|/t \ge 3/26$. These quantify the cost of maintaining low mass or a nearly legal stream.

5 Nine factors suffice for mass balance

The language threshold for a uniform mass bound differs from the threshold for eventual periodicity. Put $D(v) = 3|v| - 26(\#R(v) - \#L(v))$. Every finite physical nine-legal word satisfies

$$D(v) \le 1004.$$

Consequently every complete finite-support orbit interval satisfies

$$3(t-s) - 26(|B_t| - |B_s|) \le 1033 N_9(s,t) + 1004,$$

where N_9 counts forbidden nine-windows wholly contained in the interval. The constant 1004 is not claimed optimal. The stronger ten-language constant 322 remains valid on its narrower class.

5.1 Finite certificate and all-length coverage

Two complete enumerators produce all 164,283 feasible nine-legal words of length 150. They also cover all shorter words, whose maximum deficit is 322, including words with no legal length-150 extension. The 149-overlap graph has 157,114 vertices. Every longer real word is its initial history followed by a graph path; the converse need not hold.

There are three cyclic components: simple cycles of lengths 104 and 24, and a component with 34,712 vertices. A label-preserving graph map sends the latter to a 1,213-state quotient: every source edge maps to an edge with the same R/L label. This property preserves all physical paths even though the quotient may admit additional paths. The refinement procedure finds the map; checking its complete edge table establishes the property used here. Fourteen explicitly checked forbidden words each demand two consecutive same-colour reads of one site and are physically impossible; Appendix D prints the words and conflict indices. The reachable product avoiding these words has 9,154 states and 9,550 edges. Every quotient state is allowed as an entry with empty suffix history, which preserves coverage even when a conflict crosses component boundaries.

An integer potential $0 \le y_u \le 306$ satisfies

$$y_u - y_v \ge \omega(c) \quad (u \xrightarrow{c} v), \qquad \omega(R) = -23, \quad \omega(L) = 29.$$

All inequalities are checked after reconstructing the potential. Telescoping bounds the deficit of every internal path by 306. The two simple-cycle total weights are 0 and -32; their maximal subpath weights are 237 and 94. Removing complete laps proves these bounds for arbitrarily many repetitions.

Give each component its internal budget and each entry its maximum 149-letter history deficit. A topological dynamic program on the complete condensation graph adds crossing-edge weights and the next internal budget. Every real path visits each component at most once. Checking all initial and cross-edge inequalities gives the total bound 1004, including both ends of the word. This establishes the all-length claim without extending a finite simulation cutoff. Finally, remove the last letter of every bad nine-window. The at most N_9+1 remaining consecutive runs are nine-legal; each removed letter costs at most 29. Additivity yields $1004(N_9 + 1) + 29N_9$ and proves the unrestricted inequality.

5.2 The sharp window threshold uses finite inputs

Let

$$u = \mathrm{LLLLRRLLRRRRLLRRLRLLRR}, \qquad d = (-3, 1),$$

and define

$$A = \{(0,-1), (1,-1), (2,-1)\},$$
$$H = \{(-1,1), (-2,1), (0,0), (-1,0), (-1,-1)\}.$$

All cyclic eight-factors of u are standard, and it contains eleven letters of each colour. For every $n \ge 0$ the complete finite source

$$B^{(n)} = A \cup \bigcup_{a \in H} \{a + jd : 0 \le j < n\}$$

has $3 + 5n$ black sites and, from the origin facing north, executes u^n. At update $22n$ it first departs from the proposed repetition.

For the all-parameter proof, partition phase positions into full translation classes modulo $\mathbb{Z}d$. In each class write a phase position as $b + a_i d$ and let $h = \max_i a_i$. The five

required black rays start at $b + hd$; their finite boundary assignments are exactly A. A phase read in epoch $j < n$ has ray index $j + a_i - h < n$, so every nonnegative index is present in the truncation and every negative index uses the same boundary assignment. Literal coupling proves the entire prefix. At epoch n, phase zero first reads the missing site nd, which is white. The fixed class table and its infinite parity intervals are checked exactly.

Thus $D(u^n) = 66n$, disproving a seed-independent constant for window eight and every smaller window. For $k \geq 9$, a k-legal word of length at least k is nine-legal; shorter words have deficit at most $29(k-1)$. Therefore a finite uniform constant exists exactly when $k \geq 9$. This threshold concerns finite-word mass deficit, whereas eventual attraction is governed by the ten-factor theorem.

6 Exact first-violation macros

The decision in Theorem D can be strengthened to return the first event itself and its complete post-event state. This requires a finite-path argument inside the nonsimple component, rather than an assumption of periodicity there.

Theorem F (exact event and finite prefixes). A complete finite field may be given as the XOR of M finite arithmetic segments whose strides belong to

$$\{(0,0), (\pm 2, \pm 2), (\pm 2, 0), (0, \pm 2), (\pm 4, 0), (0, \pm 4)\}.$$

Zero stride denotes a singleton. The first future J_{10}, or its permanent absence, is decidable in polynomial bit time in this representation. A finite answer returns its exact time and full state just after the event. One call uses at most five periodic plans and 1111 literal main-orbit updates, adding at most 1530 segment terms.

Let W_0 be the diameter in the maximum norm of the actual initial support and head. Conditional on n events existing, their post-event time satisfies

$$s_n \leq n(212W_0 + 854199) + 415308n(n-1). \tag{4}$$

Every intermediate complete state lies in the initial convex hull enlarged by $[-1959n, 1959n]^2$. The first n events and their complete encoded endpoint are computable in poly$(M + n + b)$ bit operations, with n measured numerically and b the original coordinate and length bit size.

These macros preserve finite histories exactly. Their lift counts fold starts within one event-free traversal; a fresh event begins a fresh traversal. The following proof supplies both the finite graph coverage and the arithmetic continuation needed for this stronger conclusion.

6.1 The feasible-word graph and finite-path coverage

Use the graph G from Section 3.1: its 6,605 feasible 150-letter words are edges between their 149-letter prefix and suffix vertices. There are 6,501 vertices. The same-cell alternation test used there supplies physical feasibility.

Every actual event-free continuation, once its 149-turn history is a G vertex, follows G. At most 149 nonviolating ordinary updates are needed to obtain such a vertex: afterwards the last 149 turns are wholly new, and if the next turn is legal their feasible 150-word is in the complete enumeration. If the next turn is illegal, it is returned as the event instead. This covers input time zero without inventing a periodic negative-time past.

The graph has simple $\mathcal{W}$ and $\mathcal{T}$ components of lengths 104 and 24, and one component $\mathcal{H}$ with 366 vertices and 370 internal edges. The explicit quotient of Section 3.2 maps $\mathcal{H}$ to phases $0, \dots, 101$. There is an edge $j \xrightarrow{B_j} (j+1) \bmod 102$, plus $0 \xrightarrow{L} 75$, where B is a 102-letter fold word beginning R. The short loop is $A = \mathrm{L}B_{75} \cdots B_{101}$, length 28. Their literal geometry is

$$A:\ d = (0, -4),\ \text{heading restored;} \qquad B:\ d = (1, 11),\ \text{heading reversed.}$$

Every internal $\mathcal{H}$ edge is checked against this quotient, not merely sampled. The all-parameter lemma of Section 3.3 states

$$BA^nBA^mB \text{ is physically infeasible for every } n, m \geq 0. \tag{5}$$

Its explicit consecutive same-cell/same-R conflicts and their parameter coverage are in Subsection 3.3. A finite word need not extend to an infinite legal word for (5) to apply. This is the essential finite-path rather than infinite-cycle input.

The base edge set and quotient have frozen SHA256 values in the source witness. The event runtime regenerates both and checks exact equality.

6.2 A complete bounded-fold lift

Outside $\mathcal{H}$ keep each vertex v with counter 0. Inside $\mathcal{H}$ use (v, c), $c = 0, 1, 2, 3$, except that $(v, 3)$ with quotient phase 0 is forbidden. On an internal $\mathcal{H}$ edge increase c exactly when its quotient starts the long loop, namely phase 0 with appended R. All other internal edges keep c. On entry into $\mathcal{H}$ start $c = 0$; on exit forget c. Disallow $c > 3$ and disallow arrival at phase 0 with $c = 3$.

This permits two complete B loops and a third incomplete B loop; a third *completed* loop is excluded. An initial partial loop is not charged, so no assumption is made about the path before $\mathcal{H}$ entry. Between three long-loop starts at phase zero there are powers of A, and completing the third yields (5). Thus every real finite event-free path lifts. $\mathcal{H}$ cannot be revisited after leaving it because G's SCC condensation is acyclic. Counter reset on reentry cannot therefore mask additional physical folds.

The lift may still contain impossible paths. It is an exclusion-safe overapproximation, not a claim that every lifted path is physical.

Complete reconstruction yields 7595 vertices and 7763 edges. Its cyclic SCCs are simple cycles of lengths

$$24, \quad 104, \quad 28, \quad 28, \quad 28.$$

The 28-cycles are A phases at counters 0,1,2. The 24- and 104-cycles are the prior T and standard words. No claim follows from cycle lengths alone; the words and their nonzero displacements are checked too.

Assign to each condensation component the three weights (one cyclic-component visit, one noncyclic-vertex update, period): respectively $(1, 0, P)$ on a cyclic component, and $(0, 1, 0)$ on a noncyclic singleton. Backward longest-path values are nonnegative and, over every condensation edge $C \to D$, satisfy

$$\text{rank}(C) \geq \text{weight}(C) + \text{rank}(D)$$

coordinatewise. Terminal ranks dominate their own weights. Internal simple-cycle edges keep the same rank. All 7763 lifted edges are checked. The maxima of the three ranks are

$$5, \qquad 956, \qquad 212. \tag{6}$$

Python uses the base's iterative two-pass SCC routine; the independent native checker reconstructs the lift and uses Tarjan SCCs. Sorted full edge and rank hashes agree. The source witness fixes the complete edge and rank hashes. Integer inequalities, complete reconstruction and (5), not a floating-point solver, carry the proof.

6.3 Exact physical acceleration on a simple component

At a cyclic vertex propose its internal periodic word u, of length P and nonzero displacement d, restoring heading. Let q_i be its phase read positions in the actual pose. At $q_i + kd$ the required old-field contribution is determined by

$$\begin{aligned} \text{read bit} = {} & \mathbf{1}_B(q_i + kd) \\ & \oplus \Big|\{(j, s) : q_j - q_i = sd,\ 0 \leq s \leq k, \\ & \qquad\qquad s > 0 \text{ or } (s = 0 \text{ and } j < i)\}\Big| \bmod 2. \end{aligned} \tag{7}$$

This computes the literal first mismatch by induction up to that mismatch; it is not a finite-repeat heuristic. Prior-write changes occur at finitely many epoch thresholds.

For a finite segment $a + jv$, $0 \leq j < L$, its intersection with a phase ray $q + kd$ is empty, a singleton, or a finite arithmetic interval in k. Nonparallel directions use an exact determinant. For parallel directions solve the one-dimensional integer congruence after dividing by the greatest common divisor. In the closed stride family its epoch modulus is 1 or 2. This modulus matters: the 24-mode has axis stride 2 and the 28-mode axis stride 4. Treating every parallel intersection as a full interval would be wrong.

Split epochs into $k = 2z$ and $k = 2z + 1$. Each contribution is then an ordinary finite interval in z; each write threshold is a final interval. Sort parity-change endpoints separately for the two classes, including $z = 0$ and the final infinite interval. This returns the exact earliest mismatch, or certifies none. All arithmetic is on integers of polynomial bit size.

Before a finite mismatch at $t = nP + j$, append P finite write segments for the full periods (if $n > 0$), and j singleton writes for the partial period. These are XOR contributions; every skipped write is accounted for. There are at most $2P - 1$ new terms. Shift the history vertex around its simple cycle by t mod P. Read the real cell and execute its actual update. If J_{10} is true, return the complete post-state. Otherwise the real edge must leave this simple component: the unique internal edge has the different predicted label. Continue in the lifted graph.

If no mismatch exists, the actual infinite continuation equals this cycle word. A finite field cannot realize the 24- or 28-word forever: their stabilized first-read classes require respectively two and five infinite initial black rays, as proved in the prior theorem. Hence only the standard cycle can give permanent absence. All earlier actual updates were individually checked or were legal cycle edges, so the certificate says no J_{10} at any nonnegative time, not only after its standard boundary.

A component is never revisited after exit. By (6), one call therefore uses at most five plans, at most 956 noncyclic updates, at most five cycle exits, at most 149 warm-up updates, and one final event update. A safe literal bound is 1111. The sum of visited cycle periods is at most 212, hence

$$\text{new terms} \le 149 + 956 + 2 \cdot 212 + 1 = 1530.$$

The $2P$ allocation includes a possible non-event cycle exit. No skipped write or exit update is omitted. Periodic proposals and inverse-history computations use a copy until the actual endpoint is determined.

6.4 A generic finite periodic-plan halo lemma

This lemma concerns any supplied periodic turn word of length P restoring heading with $d \neq 0$, from a complete finite source $Y = (B, p, h)$. Suppose its first mismatch is at time $kP + i$, position $e = p + q_i + kd$, with phase coordinates relative to p.

Let L be the largest positive translation threshold s with $q_j - q_i = sd$, or zero if none. Since a period has P unit moves,

$$\|q_i\|_\infty \le P, \qquad L\|d\|_\infty \le P.$$

If $k \le L$ then $\|e - p\|_\infty \le 2P$. If $k \ge L + 1$, all planned previous-write parities at phase i in epochs $k - 1$ and k have stabilized. If the stable required old-field bit is 0, mismatch means e is an original black cell. If it is 1, the preceding occurrence matched, so $e - d$ is an original black cell. Thus in every case

$$e \in \operatorname{conv}(B \cup \{p\}) + [-2P, 2P]^2. \tag{8}$$

Here B is the complete source field before the plan, not an unknown active subset. For $k \geq 1$, any earlier phase point $q_j + \ell d$ differs from $(\ell/k)(q_i + kd)$ by at most $2P$ in infinity norm. For $k = 0$, all reads are within P of p. Therefore the entire finite planned read path through e is inside

$$\operatorname{conv}(B \cup \{p\}) + [-4P, 4P]^2. \tag{9}$$

Its ensuing literal update enlarges this by at most one. Unchanged source cells remain present outside the read footprint; (9) never authorizes deleting them.

For radius $R = \max_{b \in B} \|b - p\|_\infty$ (zero for empty B), (8)'s more precise alternatives give: either $k \leq L \leq P$, or $k\|d\|_\infty \leq R + \|d\|_\infty + P$. Therefore the first mismatch time is strictly less than

$$P(R + P + 2). \tag{10}$$

These are conditional-on-first-mismatch bounds. A valid infinite plan has no finite halo bound.

6.5 All-input J10 geometry, time and bit complexity

Sum (9) over all finite periodic portions of one event call. The period sum is at most 212; all actual literal updates number at most 1111. Convex Minkowski enlargements compose, giving

$$H = 4 \cdot 212 + 1111 = 1959.$$

Every intermediate complete state lies in the initial complete convex hull enlarged by H; in particular its width is at most $W_0 + 2H$ and its radius about its current head at most that width. Applying (10) to every periodic part, using $P \leq 104$, and including all literal updates, gives post-event duration at most

$$212(W_0 + 2H + 106) + 1111 = 212W_0 + 854199. \tag{11}$$

After $n - 1$ event calls width is at most $W_0 + 2H(n - 1)$. Summing (11) gives (4), since $212H = 415308$.

The support of every newly appended segment is made of actual read positions. Thus all its endpoints lie in the same enlarged initial hull, even if it later cancels other terms. After n events, all coordinate and segment-length bit sizes are $O(b+\log(n+1))$; event times also have $O(b+\log(n+1))$ bits by (4). The number of terms is at most $M+1530n$. Each call performs a fixed number of ray/segment intersections, endpoint sorts and fixed-length literal or inverse operations. Exact set/list arithmetic therefore proves poly$(M + n + b)$ bit complexity, with numerical event count n as an input parameter.

Together with the already proved equivalence between $H(X)$ and finitely many J_{10} events, (4) implies that every finite-support trajectory that never reaches the standard highway, of either the nonstandard-periodic or nonperiodic type, has at least n intrinsic J_{10} events before every time T with (4)'s right side $\le T$. In particular its liminf event count divided by $\sqrt{T}$ is at least $1/\sqrt{415308}$. This counts intrinsic events; interval-contained ten-factor counts differ only in the first nine boundary positions.

This gives a necessary lower event rate for every such trajectory.

7 Geometry of finite legal intervals

The local language controls the geometry of an entire legal interval, even when its duration is unbounded. The coordinate selected in the next theorem is fixed for the whole interval.

Theorem (endpoint strip). Let $p_0, \dots, p_L$ be the head positions of a physically feasible finite word whose contained ten-factors are standard. There is an $i \in \{x, y\}$ such that

$$\min(p_{0,i}, p_{L,i}) - 43 \le p_{t,i} \le \max(p_{0,i}, p_{L,i}) + 43 \qquad (0 \le t \le L).$$

More precisely, after translating p_0 to zero, one of the four coordinate normals n satisfies $-43 \le n \cdot p_t \le n \cdot p_L + 43$ throughout.

Proof. Use the bounded-fold lift established for the event macros. Adjoining the incoming heading gives a directed graph G with 30,380 vertices and 31,052 edges. A vertex records the length-149 history, the fold count and the actual heading; an edge appends its literal turn and carries its unit displacement. Every feasible continuation after the first 149 turns lifts to this graph. The all-parameter exclusion of BA^nBA^mB supplies coverage of the fold count.

For a cyclic strongly connected component C, let $M(C) = \{n : n \cdot d(C) \ge 0\}$, where n ranges over the four coordinate normals and $d(C)$ is the displacement of its simple cycle. Give an acyclic component all four normals. The finite certificate starts a mask at every component and propagates every possible intersection along every condensation edge. All 88,024 resulting component/mask states have nonempty masks. Closure under propagation proves by induction that every finite path has a common allowed normal. An independent work-list calculation on the original graph agrees at all 90,256 vertex/mask states.

For each normal delete the cyclic components that omit it. On the resulting G_n, the certificate gives an integer potential H_n with

$$0 \le H_n \le 33, \qquad n \cdot \Delta(v, u) \ge H_n(u) - H_n(v) \quad \text{for every edge } v \longrightarrow u.$$

There are 30,714 inequalities for each normal. Telescoping bounds every subpath's retreat by 33. These inequalities and the nonempty-mask closure, rather than the method used to find the potentials, are the finite premises.

Short words of length at most 149 have coordinate radius at most 24 by complete feasible-word enumeration. For a longer word put $q = p_{149}$, v equal to its startup lift

state, and $z_t = n \cdot p_t$. The startup certificate checks, for every such state and every allowed normal,

$$\min_{t\le 149} z_t \ge -43, \quad z_{149} - H_n(v) \ge -43, \quad \max_{t\le 149}(z_t - z_{149} + H_n(v)) \le 43.$$

For $t \ge 149$, telescoping gives $z_t \ge z_{149} - H_n(v) \ge -43$ and $z_t \le z_L + 33$. For $t \le 149$, use the first startup inequality and $z_L \ge z_{149} - H_n(v)$ in the third. Short words use the sign with $n \cdot p_L \ge 0$. This proves the oriented assertion and the strip. □

Corollary (localized events). Let K be a nonempty finite set with bounding rectangle $[x_-, x_+] \times [y_-, y_+]$. Between any consecutive chain of intrinsic J_{10} events with read positions in K, all head positions lie in

$$\{x_- - 44 \le x \le x_+ + 44\} \cup \{y_- - 44 \le y \le y_+ + 44\}.$$

The conclusion includes the outgoing move of the last event and is independent of the number of events in the chain. Indeed, each intervening silent word starts one step from K and ends in K; apply the theorem to that contiguous word and include the event moves. If infinitely many events are eventually localized in K, the entire late head trajectory lies in this same cross.

This geometry complements the complete-source bounds used by event prediction. A legal interval occupies one endpoint strip while it may travel arbitrarily far along it. The folded ribbon in Appendix A.4 realizes precisely this distinction.

8 Finite-horizon rigidity and silent supply

The infinite-future attraction theorem does not apply to a legal fragment that may later violate the language. We now bound the nonstandard part of every finite ten-legal physical word, without assuming a legal continuation. For such a word w, let $c(w)$ count distinct sites whose first requested read is L. The alternating-read criterion makes this exactly the minimum initial black mass realizing w. In particular $c(w) \le m$ for a segment starting with a complete m-black field.

Theorem (one contiguous standard core). Every finite ten-legal physical word has a decomposition

$$w = u(W_r)^k v, \qquad k \ge 0, \qquad |u| + |v| \le 114c(w) + 154693,$$

where W_r is a cyclic phase of the standard word. If Δ is the total head displacement, then

$$|w| \le 52\|\Delta\|_\infty + 114c(w) + 223437. \tag{12}$$

The coefficient 52 is optimal; the other constants are sufficient bounds. The core is one contiguous part of the actual trace, and it may be empty.

8.1 Component order and transverse supply

The finite-path fold cover established earlier represents every finite legal word by at most five cyclic stages, with periods $24, 104, 28, 28, 28$. The acyclic steps number at most 956 and the total cycle lengths at most 212. Allowing the initial 149 letters, partial cycle remainders and exits leaves at most

$$F = 149 + 956 + 212 + 5 = 1322$$

literal letters outside whole powers. The complete condensation graph checks the additional order: at most one T power precedes at most one W power, followed by at most three A powers. A departed W component cannot be reentered within the same legal word. This order uses the all-parameter three-fold obstruction, so it covers arbitrarily long finite paths.

Every phase of T^k requires at least $2k$ distinct initially black read sites, and every phase of A^k requires at least $5k$. Each whole period of either mode changes black mass by $+4$. These supply counts follow by listing phases with no earlier same-cell predecessor in any translating copy. If q_i and q_j are phase positions and d the full drift, an earlier copy can hit phase i precisely when $q_j - q_i = sd$ with $s > 0$, or $s = 0$ and $j < i$, subject to the copy being present. The fixed phase tables check all rotations and starting phases.

A W power reads an eleven-level diagonal strip, whereas an A power reads a four-level axis strip. Any translated pair intersects in at most $11 \cdot 4 = 44$ cells: an axis coordinate and a diagonal coordinate determine at most one lattice site. Thus an arbitrarily long standard block can supply only 44 sites directly to each later axial block.

8.2 Complete-field cofactor accounting

Let k_T, k_W, k_1, k_2, k_3 be the whole-period counts, inserting zeros for absent stages, and put $q = m + F$, allocating all literal flips from the start. Initially $2k_T \le q$, so the T stage adds at most $4k_T \le 2q$ black cells. The mass budget before the W stage is therefore at most $q + 4k_T \le 3q$. Let D be the complete net toggle set of the W block. After that block write the actual field as $B = R \triangle D$. The cofactor R is an algebraic device, not a substituted physical state. Every cell of D continues to participate in the actual evolution. For any toggle set Z,

$$|R \triangle Z| - |R| \le |(R \triangle D) \triangle Z| - |R \triangle D| + 2|Z \cap D|.$$

This holds pointwise: outside $Z \cap D$ the mass changes agree, and at an intersection site they differ by at most two. Consequently an A^k block increases cofactor mass by at most $4k + 88$, and at most 44 of its required first-L sites can be supplied by D. The cofactor starts with the pre-W mass budget $3q$; each earlier A^{k_j} block adds at most $4k_j + 88$. Hence

$$5k_i \le 3q + 4 \sum_{j<i} k_j + 88(i-1) + 44, \qquad i = 1, 2, 3.$$

Iteration yields

$$\sum_{i=1}^{3} k_i \le \frac{453}{125} q + \frac{15004}{125}.$$

The number E of letters outside the whole standard power therefore obeys

$$E \le F + 24k_T + 28 \sum_i k_i \le \frac{14184}{125} m + \frac{3867322}{25} \le 114m + 154693.$$

Applying this bound to the minimal first-read source proves the core claim.

The common-normal certificate from the endpoint-strip argument supplies one coordinate normal n whose projection on every cyclic drift of this path is nonnegative. If the standard power occurs, its projection is 2 per period, while all literal letters together contribute at least $-F$. Hence $2k_W \le \|\Delta\|_\infty + F$. Combining this with the bound on E proves (12), since $52F = 68744$. Without a standard power, the bound on E already suffices. Standard prefixes W^k from a fixed finite seed have length $104k$, displacement norm $2k$ and bounded first-black cost; they rule out any smaller displacement coefficient.

8.3 Closed returns, lifetimes and a one-jump algorithm

A closed legal return from a complete m-black field has length at most $114m+223437$. If a cell is born black at time b and next read at $e > b$, and the word on $[b, e)$ is ten-legal, its endpoints coincide. Thus

$$e - b \le 114|B_b| + 223437.$$

The mass belongs to the birth checkpoint.

There is also an exact first-J_{10} algorithm using one standard-periodic query and at most $114m+154694$ ordinary forward updates on an explicit complete m-point source. Recover the initial nine turns by inverse dynamics on a copy of the full field. Execute literal updates until the first event, or until the actual 149-letter history enters the standard component. At that point the complete-field ray-parity query either certifies a permanent standard future or gives its first mismatch. Retain every skipped write as finite XOR segments and continue literally until the first event. The component order forbids a second standard stage before that event. The core bound limits all remaining literal work and proves totality.

The returned endpoint contains the original points, $O(m + 1)$ point changes and at most 104 finite translating segments. Coordinates and elapsed time may be arbitrarily large binary integers; the algorithm has polynomial bit complexity in the explicit sparse input. This complements the five-plan algorithm of Theorem F for compressed inputs representing huge black mass. Both algorithms retain all unread debris.

For a fixed finite target K, put $R = \max_{q \in K} \|q - p_0\|_\infty$. If the first future event lies in K, its pre-update index is at most $52R + 114m + 223437$. This bounds a first-event query for the specified initial state.

9 Limitations

The attraction theorem assumes a legal future; the mass inequalities apply to arbitrary finite trajectory intervals. Together they give local-event algorithms and necessary density laws. They provide no upper bound on the total number of events of an arbitrary finite seed. The ten-factor threshold for attraction is sharp over arbitrary initial colourings; its optimality on finite black supports is a separate question. The mass theorem has the distinct sharp window threshold nine.

The coefficient twelve in the attraction bound and the constant 322 in the ten-factor deficit bound have matching witnesses. The attraction intercept 212 and the penalty coefficient 351 are sufficient constants whose optimality is open. Algorithmic costs distinguish explicit point sets from compressed XOR segments and depend polynomially on the numerical event count, rather than its bit length. Lifetime bounds use the complete mass at the named checkpoint; repeated application does not replace it by the initial mass. The memory families have finite prescribed prefixes and recoveries. Whether one finite seed can sustain autonomous infinite renewal remains a global dynamical question.

10 Reproducibility and conclusion

The source package provides two complete finite kernels. The attraction kernel regenerates the 6,605 physical length-150 words, the quotient of the nonsimple component, and the exact path and supply bounds. The mass kernel regenerates all 29,559 length-300 edges, checks every integer potential inequality, and replays the equality witness. Python and C++ implementations are compared, and damaged certificates are rejected. The event lift additionally checks all 7763 edges and component-rank maxima 5, 956 and 212. The all-parameter foldback exclusion, seed induction and spiral separation are proved in the text; finite geometric checks verify their stated numerical premises.

The memory constructions additionally reconstruct every affine read interval, compare complete endpoint fields, verify the all-future standard recovery, and check every alignment in the reset lower bound. Their universal parameter coverage comes from the finite arrangement proof, with separate literal simulations and damaged-witness controls.

The silent-supply checker independently rebuilds the 6,605-word cover, component order and common normal, checks every phase supply and transverse intersection bound, and compares 174 complete first-event endpoints. The nine-factor checker independently enumerates all 164,283 length-150 words, checks 9,550 product-edge inequalities and the complete condensation budget, and verifies the all-parameter eight-factor truncation.

Run `python3 factor_language/reproduce.py` from the source root. Each checker implements the finite coverage or integer inequality stated in its corresponding proof, and records its source identifiers and verification results. The analytic arguments establish coverage for all parameters; the exact computations establish their finite premises.

The computational sources and certificates are fixed at repository commit e932522, factor project. The `sources.json` manifest lists their hashes. The accompanying source archive includes those finite kernels and this article's source.

The two proof routes expose complementary structure: a legal infinite future has rigid eventual motion, while an arbitrary finite interval must pay for its mass deficit with local violations. Their combination connects a local signature of P104 with both a decidable event and a quantitative all-input law.

A Memory produced during legal intervals

A finite ten-legal interval can perform substantial computation on its complete input field. We give an explicit family that manufactures black lifetimes later consumed at violations, complete its autonomous recovery, and then study a related family with a quantitative reset cost. These constructions identify the spatial information used by the event macro.

A.1 A legal producer and a finite consumer

Use the words A, B of the two-loop component and set

$$V_n = B[9:]A^nBA^{n-1}, \qquad P = \text{RLRRRRLRRRRLLLLRLRRL}, \qquad n \geq 1.$$

Here $B[9:]$ removes its first nine letters. For finite templates L_0, M_0, H_0, define

$$B_n = L_0 \cup \bigcup_{j=0}^{n-1} (M_0 + (0, 4j)) \cup (H_0 + (0, 4n)).$$

The complete templates are

```
L0 = {(-1,0),(-1,3),(-1,6),(-1,7),(0,-1),(0,3),(0,5),(0,7),
      (1,-1),(1,2),(1,3),(1,4),(1,6),(1,7),(2,-2),(2,2),
      (2,4),(2,5),(3,-2),(3,1),(3,2),(4,-1),(4,0)}
M0 = {(-1,10),(-1,11),(0,8),(0,9),(0,11),
      (1,8),(1,10),(1,11),(2,8),(2,9)}
H0 = {(-3,11),(-3,12),(-2,9),(-2,10),(-2,13),
      (-1,9),(-1,11),(-1,12),(-1,13),(0,8),(0,9),(0,12),
      (1,8),(1,10),(1,12),(2,8),(2,9),(2,12)}
```

Their vertical ranges are disjoint. Start the ant at $(-1, 0)$, heading west, with precisely B_n black; its mass is $10n + 41$.

Theorem (silent production and later consumption). The trajectory begins with V_nP^{n+1}RLL. Put $T_n = 56n + 167$. There is no intrinsic J_{10} at $0 \leq t \leq T_n$; the first occurs at $T_n + 1$, at $(1, 4)$, heading east, and leaves the head at $(1, 5)$, heading north. The largest visited height before that event is $4n + 13$. Every initial black lifetime has already been consumed before T_n.

For $0 \leq j \leq n$, the following are consecutive white/black reads:

site	birth	consumption	age
$(1, 4 + 4j)$	$44 + 28j$	$56n + 168 + 20j$	$56n + 124 - 8j$
$(0, 7 + 4j)$	$56n + 146 - 28j$	$56n + 186 + 20j$	$40 + 48j$

Every birth is in the J_{10}-free interval and every displayed consumption is a J_{10}. There are exactly $9n + 6$ violations during P^{n+1}. The first mismatch with the continuation P^{ω} is at relative time $20n + 22$ after T_n.

Proof by a complete affine arrangement. Every phase of a variable block has a fixed horizontal coordinate, vertical coordinate $4an+b+4\epsilon k$, and repetition index $0 \le k < cn + d$, where $\epsilon = \pm 1$. In a physical cell class $(x, 4r + s)$, its visits therefore occupy a half-open interval

$$\alpha n + \beta \le r < \gamma n + \delta.$$

Their times are affine in n, r. Within a repeated block the coefficients are the same for every phase, so the constant term fixes chronological order; distinct blocks retain their script order. Literal phases and input points give singleton intervals. All 274 phase records are represented.

Collect the affine endpoints of source cells, visits, target cells and the selected lifetimes. Their order changes only at pairwise rational crossings. Splitting at the floor and successor of every crossing produces the complete parameter partition $n = 1, 2, 3$ and $n \ge 4$. Within each slab, an affine inequality is checked at its endpoints; the unbounded slab additionally requires the appropriate slope sign. Between consecutive spatial endpoints, the source bit and entire ordered read list are constant. Starting at the source bit and toggling after each read checks physical feasibility for every cell and every parameter in that slab. Including all target intervals also checks the complete endpoint field, including sites not read by a block.

The declarative witness records complete low/bulk/high endpoint profiles. The verified endpoint data are

prefix	time	pose	black mass
V_n	$56n + 167$	$(0, 4)$, N	$18n + 58$
$V_n P^{n+1}$	$76n + 187$	$(0, 4n + 8)$, N	$22n + 62$
$V_n P^{n+1}$RLL	$76n + 190$	$(0, 4n + 9)$, W	$22n + 61$

For every initially black band, the arrangement also verifies a read in V_n, proving exhaustion of source lifetimes. On each selected lifetime band, the last producer read is R and the first consumer read is L; their affine times give the table above. Endpoint-and-slope identities cover the entire band, including all intervening integer cells. There are 607 nonempty affine bands and 27 selected lifetime bands in the exact certificate.

Nine inverse updates stay below height eight and recover the true prefix $B[: 9]$. All ten-factors of A, B and their possible junctions occur in $BAABAA$; its 307 windows are legal. The final nine letters of A and B agree, so the producer's outgoing context is independent of n. With that context, violations in the first P occur at indices $1, 2, 3, 4, 5, 19$, and in later copies at $0, 1, 2, 3, 4, 5, 6, 7, 19$. This proves the first event and $6 + 9n$ count. The affine path endpoints give the exact maximum height. Finally the following RLL differs from P at its third letter, giving the exact mismatch time. □

The construction supplies $2(n+1)$ distinct black lifetimes produced without a violation and later consumed at violations. At least $n + 1$ have age at least $48n + 124$. The cell $(1, 4)$ is born at the fixed time 44 and survives until the first violation, with age

$56n + 124$. For any fixed D, let S_D mean a black read at a site absent from the preceding D actual visits. Those sufficiently old consumptions are simultaneously J_{10} and S_D events, with these two predicates testing the turn vocabulary and visit recency respectively.

The fixed source and event poses accompany an unbounded excursion. Thus a first-J_{10} macro requires information about its complete source geometry; its two endpoint poses alone do not bound its path or duration. The earlier complete-source convex-hull bound provides exactly that geometry. The black mass of this family grows with n.

A.2 The consumer's finite fuel budget

Each copy of P restores the heading and moves by $(0, 4)$. Its entire read footprint has local heights zero through three, so consecutive copies have disjoint read footprints. The first-L cells are $(1, 0)$ and $(0, 3)$, and both must already be black at the beginning of the train. Consequently a complete field of mass m supports at most $\lfloor m/2 \rfloor$ consecutive whole copies of P.

One copy leaves four net new black cells, but these lie behind the next copy's required first reads. The exact distinction between total mass and remaining fuel explains how the constructed finite consumer can increase black mass while exhausting the input it needs to continue.

A.3 Complete recovery of the producer family

The same $10n + 41$-black field B_n admits an exact infinite continuation. Thus the producer, its later consumer and the eventual highway can be studied on one complete autonomous trajectory.

Theorem (complete archive recovery). Every B_n, $n \geq 1$, eventually enters standard P104. Let τ_n be its first permanent entry and E_n its total number of future intrinsic J_{10} events, including the nine windows across entry. Then

n	τ_n	E_n	$\lvert B_{\tau_n} \rvert$
$n \geq 10$ even	$550n + 6714$	$507n/2 + 3154$	$50n + 337$
$n \geq 17$ odd	$712n + 11087$	$(743n + 10351)/2$	$46n + 622$

The corresponding entry poses are $(-10, -3)$, W and $(16, 8)$, N, with drifts $(-2, -2)$ and $(2, 2)$. In both infinite classes the last intrinsic event is at $\tau_n + 8$. The remaining cases are

n	1	2	3	4	5	6
τ_n	7936	6564	24214	5244	16825	15026
n	7	8	9	11	13	15
τ_n	11134	12786	18441	17125	21371	21603

Proof. At $s_n = 76n+190$ the verified producer–consumer prefix ends at $(0, 4n+9)$, W with complete mass $22n+61$. The supplied grammar continues this exact state with 21

variable loops and 22 fixed words in the even regime, or 27 variable loops and 28 fixed words in the odd regime. The loop drifts are vertical, of magnitude four or eight; their lengths and affine repeat counts are derived from their stored letters. Summing them gives additional durations $474n+6524$ and $636n+10897$. The source archive contains these literal words in `verification/silent_archive_recovery/grammar.json.zlib`; `unpack_grammar.py` reconstructs the JSON description and `grammar-manifest.json` records each word's length and hash. The complete word data are part of the computational certificate; the following argument explains why its finite comparisons cover every parameter.

For execution, write $n = 2k + 10$ or $2k + 17$. Split displacement-four loop indices by parity and represent physical sites as $(x, \rho + 8r)$, $0 \le \rho < 8$. Each phase becomes a record with an affine integer interval $L(k) \le r < U(k)$ and an affine time $ak+br+c$. Input cells have the same representation. All source cells and all reads are included. Endpoint crossings split the even domain into $k = 0$ and $k \ge 1$, and the odd domain into $k = 0$, $k = 1$ and $k \ge 2$. These partitions are derived by comparing all affine endpoints. Endpoint tests, together with the slope on an unbounded slab, establish every order inequality over its entire domain.

Within each spatial band the source colour and the ordered list of reads are fixed. The checker verifies strict chronology and alternates the actual colour at each read. This covers 11,371 spatial bands, 91,849 read checks and 80,982 chronological inequalities. Induction over time proves the whole script for every parameter. Source XOR prefix reads gives its complete terminal field, including unread cells.

All pre-tail black cells have x coordinates in $[-24, 15]$ or $[-24, 20]$. For standard phase positions q_i and drift d, the colour at $q_i + ed$ is the terminal membership bit XOR the parity of all earlier tail visits. Since $d_x = \pm 2$, only finitely many epochs can meet the pre-tail support. Checking their membership-change thresholds, every self-visit threshold and the final infinite interval reduces the entire tail to 1,255 all-parameter parity checks. Induction then proves permanent standard motion. The last pre-entry letter differs from the backward extension of the uniquely aligned standard phase, proving that this entry is first.

Nine inverse updates supply the true initial context. For each variable loop, after two repetitions its outgoing nine-letter context is fixed, so further copies add a fixed number of violations. Counting the exceptional first copies and the nine entry windows gives E_n and the last-event time. Mass follows independently by summing right minus left turns. Finally, the twelve excluded values of n have literal scripts, complete endpoint checks and the same infinite-ray verification. They exhaust the complement of the two infinite regimes. □

The certificate also gives exact first-read queries: minimize the affine read times in the finitely many records and the 104 terminal rays. The number of records is fixed independently of n, so distant queries preserve the complete field without expanding the intervening trajectory.

A.4 Autonomous recovery of a folded ribbon

A second family gives a full infinite continuation, providing a separate reset-cost comparison with the producer family. Define

$$\begin{aligned} C_0 &= \text{RLRLLRLLRLRLLRRRRLLRLLRRRRLLRRRRLLRLRRRRLRLL}, \\ D_0 &= \text{LLRRRRLRRLRRRRLLLLRLRRRRLRL}, \\ w_n &= C_0 A^{n+6} B A^{n+2} D_0, \qquad |w_n| = 56n + 397, \quad n \geq 0. \end{aligned}$$

Run the geometric plan from the origin facing north, and place initial black cells precisely at sites whose first requested letter is L. All other cells are white. This specifies a complete seed X_n with $10n + 76$ black cells.

Theorem (recovery and reset cost). The prescribed prefix is physically feasible. It writes the origin black at time zero and next reads it at time $56n + 396$. Its intrinsic J_{10} events occur at times zero through eight, followed by none until time $56n + 397$. Its first permanent standard time and complete future violation count are

$$\tau(X_n) = 152n + 2527, \qquad \#\{t \geq 0 : J_{10}(F^t X_n)\} = 56n + 926.$$

Let $Y_n = F^{56n+398} X_n$. A literal repair of length ℓ means an actual ant trajectory $Y_n \mapsto F^\ell Y_n$, with no external edits. Any such repair from Y_n enabling the prefix w_m, at any translated and quarter-turned pose and with arbitrary unread debris, satisfies

$$\ell \geq \max(0, 4m - 136).$$

Proof. The exact recovery is six alternating scans,

$$E_0 V_1^{n+5} E_1 V_1^{n+5} E_2 V_2^{n+4} E_3 V_2^{n+4} E_4 V_3^{n+3} E_5 V_3^{n+4} E_6,$$

where

$$\begin{gathered} V_1 = \text{RLRLLRLR}, \quad V_2 = \text{RLRLLRLLLLRRRRLR}, \\ V_3 = \text{RLRLRLLLLRLRRRRLLLLRLRRR}. \end{gathered}$$

The seven fixed words E_i, of lengths $3, 258, 2, 142, 449, 566, 542$, are the literal finite data in the supplied `verification/silent_fold_reset/grammar.json`. Their geometric phases and all variable scans are checked from their letters. This recovery has length $96n + 2338$, ends at $(8, 7)$, east, with mass $18n + 301$, and preserves every historical cell.

For these words, writing a cell as $(x, r - 4k)$ gives affine visit intervals whose endpoint slopes are zero or one. All pairwise endpoint crossings occur by parameter one. Consequently the complete order cases $n = 0, 1, 2$ cover every $n \geq 0$; equality types persist beyond the final crossing. The 2,287 derived phase records check all requested colours in chronological order and give exact complete fields. The full reduction and recovered cap coordinates are included in the supplemental proof and mask certificate.

The final 208 letters are two copies of one standard phase, preceded by a letter disagreeing with its backward extension. At that endpoint, all black cells have horizontal coordinates between -14 and 10. Each future standard ray of drift $(2, 2)$ therefore

has only finitely many possible source hits. Checking the source hits, self-write overlap thresholds, and final infinite interval certifies the entire tail. The order-case argument covers all their colours for every n. Subtracting 208 from the endpoint time gives the stated first entry. Each added pair of scans contributes $2(8+9+11) = 56$ intrinsic violations; the exact zero-parameter count is 926, including the initial inverse-history windows. The last violation is at $\tau(X_n) + 7$.

For the reset bound, the source requirements I and output bulk O on a four-row tile are

x	I	O
-5	00..	1100
-4	0000	1111
-3	0011	1100
-2	1101	0001
-1	1011	1000
0	1100	0011
1	0000	1111
2	..00	0011

A dot imposes no requirement. The exact overlap table checks every vertical residue, horizontal shift and parallel orientation, giving at least four mismatches per complete target tile. Two boundary tiles can lose at most eight from this lower bound; the 128 output cap cells can repair at most 128 further mismatches. This gives $4m - 136$. At perpendicular orientations, the width-eight source meets at most three target tiles; the remaining tiles each require ten black cells against white, giving $10 \max(0, m - 3) \geq 4m - 136$. Every repair update toggles one site, so it needs at least as many updates as distinct necessary corrections. □

For a clean reset to exactly the complete seed of size $10m + 76$, the output mass change from Y_n is $10m - 18n - 52$. The mass–defect inequality of Theorem E consequently gives

$$351N_{10}(0, \ell) \geq 3\ell + 468n - 260m + 1030.$$

Here $N_{10}(0, \ell)$ counts forbidden ten-windows wholly contained in the ℓ-step repair. For $m = n \geq 34$, the flip lower bound yields $N_{10}(0, \ell) \geq \lceil (220n + 622)/351 \rceil$. This last event bound concerns a clean reset; the flip bound itself permits arbitrary unread debris. Neither statement assumes that such a reset exists.

B Exact words

The machine-readable words are in `verification/ten/certificates/witness.json`; line breaks below are not letters. All geometric statements use north at the origin unless a phase placement is explicit.

B.0.1 W (length 104)

```
RRLLLLRLRRRRLRRRRLLLLRLRRRRLRLLRRLLLLRRLLRRRRLLRRLRL
LRLLRLRLLRRRRLLRLLRRRRLLRRRRLLRLRRRRLRLLLLRRRRLRRLRR
```

B.0.2 U (length 28)

RRLRLLLLRRRRLRRLRRRRLLLLRLRR

B.0.3 T (length 24)

RRRRLLRLLRRRRLLRRRRLLRLL

B.0.4 A (length 28)

LLRRRRLRRLRRRRLLLLRLRRRRLRLL

B.0.5 B (length 102)

RRLLLLRRLLRRRRLLRRLRLLRLLRLRLLRRRRLLRLRRRRLRLLLLRRRR
LRRLRRRRLLLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLL

B.0.6 C (length 89)

RRLRRRRLLLLRLRRRRLRRLRRRRLLLLRLRRRRLRLLLLRRRRLRRLRRR
RLLLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRR

B.0.7 Z (length 104)

RRRRLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRRRRLLLLRLR
RRRLRLLRRLLLLRRLLRRRRLLRRLRLLRLLRLRLLRRRRLLRLLRRRRLL

C First-black supply table for the 24-mode

The two rays are each advanced by 24. The finite times and the rays are disjoint as read events and correspond to distinct first-read cells.

Phase	Finite first-L times	Ray starts	Maximum $L - 12b_j(L)$
0	5,7	8,23	5
1	3,4,6	7,22	3
2	2,3,5	6,21	2
3	1,2,4	5,20	1
4	0,1,3	4,19	0
5	0,2	3,18	0
6	1	2,17	1
7	0,7	1,16	0
8	6	0,15	0
9	5	14,23	5
10	3,4	13,22	3
11	2,3,11	12,21	2
12	1,2,10,19	11,20	1
13	0,1,7,9,18	10,19	0
14	0,6,8,17	9,18	0
15	5,7,16	8,17	5
16	3,4,6,15	7,16	3
17	2,3,5,14	6,15	2
18	1,2,4,13	5,14	1
19	0,1,3,12	4,13	0
20	0,2,11	3,12	0
21	1,10	2,11	1
22	0,7,9	1,10	0
23	6,8	0,9	0

D Forbidden words for the nine-factor certificate

The fourteen exclusions used in Section 5 are the seven words below and their letter-wise complements (interchange R and L). For each word the table gives consecutive visits to the same cell demanding the same turn. Indices start at zero; the geometric walk starts at the origin facing north. This is a direct finite-word obstruction, valid on every initial colouring. The listed conflicts have been checked with both integer coordinates and complex-coordinate rotations.

Word	Length	Earlier	Later	Cell	Turn
1	161	0	160	$(0, 0)$	R
2	161	0	160	$(0, 0)$	L
3	153	0	152	$(0, 0)$	R
4	153	0	152	$(0, 0)$	L
5	253	0	252	$(0, 0)$	L
6	253	0	252	$(0, 0)$	R
7	181	0	180	$(0, 0)$	R
8	181	0	180	$(0, 0)$	L
9	217	0	216	$(0, 0)$	R
10	217	0	216	$(0, 0)$	L
11	309	0	308	$(0, 0)$	L
12	309	0	308	$(0, 0)$	R
13	237	0	236	$(0, 0)$	R
14	237	0	236	$(0, 0)$	L

Word 1 (word 2 is its letterwise complement):
RRLRLLRLLRLRLLRRRRLLRLRRRRLRRLRRRRLLLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRR
LRLLRRLLLLRRLLRRRRLLRRLRLLRLLRLRLLRRRRLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRR
RRLRLLLLRRRRLRRLR

Word 3 (word 4 is its letterwise complement):
RRLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLRR
LLLLRRLLRRRRLLRRLRLLRLLRLRLLRRRRLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLL
LLRRRRLRR

Word 5 (word 6 is its letterwise complement):
LRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLRRLLL
LRRLLRRRRLLRRLRLLRLLRLRLLRRRRLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLLLR
RRRLRRLRRRRLLLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLRRLLLLRRLLRRRRLLRRL
RLLRLLLRLRLLRRRRLLRLRRRRLRLLLLRRRRLRRL

Word 7 (word 8 is its letterwise complement):
RRLRRRRLLLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRR
RLRLLRRLLLLRRLLRRRRLLRRLRLLRLLRLRLLRRRRLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLR
RRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLRR

Word 9 (word 10 is its letterwise complement):
RRLRLLRLLRLRLLRRRRLLRLRRRRLRRLRRRRLLLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRR
LRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLRRLLLLRRLLRRRRLLRRLRLLRLLRLRLLRRRRLLRLRR
RRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLLLRRRRLRRL
R

Word 11 (word 12 is its letterwise complement):
LRRLRRRRLLLLRLRRRRLRLLRRLLLLRRLLRRRRLLRRLRLLRLLRLRLLRRRRLLRLRRRRLRLLLLRR
RRLRRLRRRRLLLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRL
RRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLRRLLLLRRLLRRRRLLRRLRLLRLLRLRLLRRRRLL
RLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLLLRRRR
LRRLRRRRLLLLRLRRRRLRL

Word 13 (word 14 is its letterwise complement):
RRLRRRRLLLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRR
RLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLRRLLLLRRLLRRRRLLRRLRLLRLLRLRLLRRRRLLRLR
RRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRLLLLRRRRLRR
LRRRRLLLLRLRRRRLRLLRR

Notes on sources

For reference [3], Theorem 3.1 and Remark 3.2 supply full-drift visit chronology and seed reconstruction.

For reference [4]: Author manuscript (2005), Section 6, Proposition 8 and Theorems 1–2: Universidad de Concepción preprint pp05-10.

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