Uniqueness of the Classical Langton Ant Highway from Its Layer Profile

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Abstract

Can the trail left by a periodic Langton ant determine its entire turn sequence? We prove a uniqueness theorem from sparse geometric information about that trail. Consider a periodic trace generated from a finite black seed whose period restores the heading, translates by $(2, -2)$, increases the number of black cells by twelve, and has transverse coordinate $x + y$ spanning ten units. On each interior level of $x + y$, suppose at most one of the two residue classes modulo this translation is visited an odd number of times per period. Then the period is 104 and the turn sequence is the standard highway, up to cyclic phase. The locations of the odd classes and the period length are derived from these assumptions. The proof first uses parity to restrict the possible layer profiles, then obtains a finite period bound from an integer certificate, and finally exhausts the remaining traces with their chronological visit order preserved. A second part treats the inverse problem with a prescribed wake: for every non-axis drift, a computable period bound makes synthesis terminate. Additional parity tests extend to general non-axis drifts and give a purely analytic exclusion of unit-diagonal full drift.

Keywords. Langton's ant; finite support; periodic highway; computer-assisted proof; wake rigidity; exact enumeration.

1 Introduction and main theorem

A periodically translating ant leaves a black trail behind it. Its turn word determines that trail, but the reverse question is less clear: how much of the trail determines the word? We prove that a sparse description of the layers of the classical highway already forces its full 104-turn sequence.

The classical ant turns right on white and left on black, toggles the current cell, and moves one lattice edge. We use Cartesian headings $\mathrm{N} = (0, 1)$, $\mathrm{E} = (1, 0)$, $\mathrm{S} = (0, -1)$ and $\mathrm{W} = (-1, 0)$, numbered $0, 1, 2, 3$ in that order. A complete finite seed lists every initially black cell; all other cells are initially white. A periodic trace repeats a turn word while restoring the heading and translating the head by a nonzero vector. Even when the initial black seed is finite, this trace leaves an unbounded wake. Cells related by integer multiples of the period's displacement belong to one *translation class*. Classes receiving an odd number of visits per period describe the black strands of the wake; classes receiving an even number contribute no persistent strand.

For displacement $(2, -2)$, each diagonal level $x + y = t$ has two translation classes, distinguished by whether x is even or odd. We count how many of those two classes are odd, rather than specifying which ones they are. A previous classification prescribed every odd class of the standard wake [5]. Here only the displacement, black growth, transverse width and a sparsity condition on the interior levels are given. The complete wake and period are consequences of these data.

The next definition makes the sparsity condition precise. Transverse width measures the range of $x + y$ visited by the trace during a period.

Definition 1.1. Fix an initial position and heading. Executing the geometric turns of a heading-restoring word $w = w_0 \cdots w_{P-1}$ gives pre-update positions $q_i \in \mathbb{Z}^2$ and $d = q_P - q_0$ its drift. For $d = (2, -2)$ the full classes modulo $\mathbb{Z}d$ have labels

$$(t, r) = (x + y, x \bmod 2).$$

Let o_t be the number of classes at level t visited an odd number of times in the period. Let $g = \#\mathrm{R} - \#\mathrm{L}$ and

$$W = \max_i(q_{i,x} + q_{i,y}) - \min_i(q_{i,x} + q_{i,y}).$$

A right turn adds one black cell and a left turn removes one, so g is the net black-cell increase over the period. The wake is *interior-thin* when $o_t \leq 1$ at each level strictly between the two extrema.

Thus an interior level may be visited many times; thinness requires only that at most one of its two residue classes have an odd total visit count. For example, the standard profile has counts $(2, 1, 1, 1, 1, 1, 1, 0, 1, 1, 2)$ from its lowest to its highest level. Our hypotheses allow the location of the empty level and the occupied interior residues to vary.

Theorem 1.2 (Thin-wake rigidity)**.** *Let a word w repeat forever under the classical Langton rule from a complete finite black seed. Suppose it restores the heading, has full drift $(2, -2)$, has net black growth $g = 12$, and has transverse width $W = 10$. If its wake is interior-thin, then $P = 104$ and w is a cyclic phase of the standard highway.*

There is no period bound in this hypothesis. The seed need not occupy a prescribed box, and genuinely unread stationary debris is allowed. The theorem also applies to an eventual periodic suffix after an arbitrary finite transient. Its boundary state is again a complete finite-support state.

Corollary 1.3. *Any nonstandard finite-support periodic highway with drift $(2, -2)$, growth twelve, and transverse width ten has both longitudinal residue classes odd on at least one strictly interior transverse level.*

The full displacement is measured over the chosen word. For example, repeating the standard word twice gives displacement $(4, -4)$ and growth 24. Fixing $(2, -2)$ therefore fixes the scale of the period in the classification.

Jillhewar's transverse exclusions already give width at least ten for every diagonal finite-support periodic highway [2, Section 9, theorems "No highway of transverse width two, four, six, eight"]. Theorem 1.2 addresses uniqueness within that smallest surviving width, with the additional full-drift, growth and interior-thinness hypotheses. The general inverse problem in Appendix A starts instead from a prescribed wake and non-axis drift and computes a terminating search range.

The proof combines analytical reductions with finite integer computations. The supplied-word realization criterion is prior work [2, Section 3, "Periodic-realisability criterion"]; short-factor constraints belong to the symbolic study of the ant [1]. The specific upper certificate and ordered-event engine are retained from the standard-wake project [5]. The contribution here is the removal of the complete-wake and missing-level hypotheses, together with the complete classification of the resulting finite family. No lower-width exclusion is used as a premise. Appendix B develops a separate necessary parity test for arbitrary-width mixed wakes, extending the attributed port-permutation factorization.

2 Finite support and stabilized visit order

A complete state is $X = (B, p, h)$, where $B \subset \mathbb{Z}^2$ is finite and $h \in \{N, E, S, W\}$. Write F for one literal update and $\chi(x, y) = (-1)^{x+y}$. The quantity

$$\sigma(X) = \begin{cases} \chi(p), & h = N, S, \\ -\chi(p), & h = E, W \end{cases}$$

is invariant: each update changes both position parity and heading axis. All times below are pre-update times.

Fix a nonzero drift and write the positions in one class as $q_i = b + a_i d$. A visit to the physical cell $b + nd$ at phase i occurs, when present, at time

$$(n - a_i)P + i.$$

Thus the stabilized order of phases in that class is decreasing a_i, followed by increasing i. Equivalently, its keys are $i - Pa_i$.

Lemma 2.1 (Supplied-word criterion)**.** *A heading-restoring nonzero-drift word admits a finite-support realization if and only if every class, in stabilized order, alternates* R, L, R, L, ... *starting with* R.

Proof. Far enough along a class every phase is present, and infinitely many such physical cells are initially white. Their first turn is R; successive reads toggle the colour. This proves necessity. For sufficiency, the phases available at $b+nd$ form the suffix obtained by removing all phases with $a_i > n$. For $\min a_i \le n < \max a_i$, initially blacken this cell precisely when the number of removed phases is odd; leave every other cell white. Each available suffix starts with the correct colour and alternates. Induction in real time verifies every prescribed update. Only finitely many boundary cells have been blackened. This is the construction of [2, Section 3, "Periodic-realisability criterion"]. □

Each class therefore contributes zero or one to #R − #L, according as its cardinality is even or odd. In particular,

$$g = \sum_t o_t. \tag{1}$$

Lemma 2.2 (Extreme classes)**.** *Every visited class on a transverse extreme has exactly one event, a right turn. In the top-rail normalization the top is $t = 0$, the head faces east, $\sigma = -1$, and the bottom is $t = -10$.*

Proof. The E and N steps increase $t = x + y$; W and S decrease it. Since $\chi = (-1)^t$ is fixed on a level, the invariant σ fixes the arrival axis there. At the top, the incoming heading is N or E and the outgoing heading is S or W. Hence every visit to a fixed extreme class has the same arrival and departure directions, and thus the same turn. Lemma 2.1 forces this constant stabilized word to be the singleton R. At the top, N followed by a right turn would move E above the extreme; the admissible right turn is E to S. At the bottom it is W to N. Translate a top event to the origin. The stated gauge and width follow. □

There are at most two extreme classes on either boundary, because the quotient is by the full vector $(2, -2)$, not its primitive direction. The geometric interval is a consequence of the candidate's stated width and normalization; it is not a wall added to the original dynamics.

3 Five possible profiles, derived analytically

Let m_t count the phase events on level t. Let u_t count straight passages from $t-1$ through t to $t+1$. Flow balance in the closed transverse walk gives the same number of reverse straight passages. At even t, right turns reverse the transverse direction and left turns continue it; at odd t the roles are reversed. Consequently

$$m_t = (-1)^t o_t + 4u_t. \tag{2}$$

The movement axis alternates, so there are $P/2$ horizontal moves. Their net displacement is two, implying that $P/2$ is even. Summing (2) gives

$$P \equiv 0 \pmod 4, \qquad \sum_t (-1)^t o_t \equiv 0 \pmod 4. \tag{3}$$

Proposition 3.1 (Profile reduction)**.** *Under Theorem 1.2's hypotheses both extreme levels have two odd classes. Exactly one interior level is empty, and it is one of*

$$h \in \{-9, -7, -5, -3, -1\}.$$

All other interior levels have one odd class.

Proof. The two rails supply at most four odd classes; the nine interior levels supply at most nine. Growth twelve forces the rail total to be three or four. If it is three, all nine interior levels are occupied. Their signed sum is $4 - 5 = -1$, so the total in (3) is two, a contradiction. Thus the rail total is four and exactly eight interior levels are occupied. If h is the empty level, the signed sum is $4 + (4 - 5) - (-1)^h = 3 - (-1)^h$. It is divisible by four exactly for odd h. □

For a fixed h, list the eight occupied interior levels in increasing order. An eight-bit mask assigns their longitudinal residues. There are 256 masks. Changing the origin to the other singleton top event swaps every residue, i.e. replaces the mask by its bitwise complement. We choose the origin so that the highest occupied interior strand has residue one, giving masks $128, \ldots, 255$. This normalization uses an actual top event and a lattice translation; it uses neither reflection nor colour complementation.

Corollary 3.2 (Standard layer profile)**.** *Suppose the full drift is* $(2, -2)$ *and, after normalizing the highest odd level to zero, the odd-class counts at levels* $-10, \ldots, 0$ *are*

$$(2, 1, 1, 1, 1, 1, 1, 0, 1, 1, 2),$$

with no odd classes elsewhere. Every finite-support periodic realization has period 104 *and the standard read word. The longitudinal residues on the eight singleton levels need not be specified.*

Proof. The supplied-word criterion gives growth twelve. Every visited extreme is odd, by Lemma 2.2, so the actual width is ten. The supplied profile is interior-thin. Apply Theorem 1.2. □

In particular, specifying all twelve standard class labels gives the complete-wake uniqueness theorem [5]. There are two proof routes from the five profiles. The direct route in Sections 4–7 checks all 76 residue assignments after count reduction. The mixed-wake parity identity of Appendix B first eliminates the profiles with empty level $-9, -5, -1$, leaving 40 assignments; Section 7.1 explains that shorter route. We present the full enumeration because its proof uses only local count and chronological conditions, independently of the port-permutation identity.

4 An all-period upper certificate

The period is bounded before any chronology search. In the top-rail normalization $\sigma = -1$, the analytic clock bound of Theorem A.2 gives

$$P \le \sum_{t=-10}^{0} (-1)^t o_t (3^{|t+5|} - 1) \le 928$$

for each of the five profiles. This already makes the inverse problem finite. The following integer graph certificate reduces the caps to 160–192, which makes exhaustive enumeration practical.

Let $\mathcal{B}$ be the supplied list of 588 impossible words, of length at most 21. Each has consecutive reads of a physical cell requesting the same turn, so it is impossible on every initial colouring. Construct contexts from the empty word and all prefixes of words in $\mathcal{B}$, retaining those with no suffix in $\mathcal{B}$. On appending a letter, reject a forbidden suffix; otherwise retain the longest suffix that is a context. This defines a deterministic suffix automaton with 4,931 contexts.

A graph vertex is (t, h, a) with $-10 \le t \le 0$ and $h \equiv t + 1 \pmod 2$, where a is a context. Place the geometric walk of a so that its endpoint has transverse coordinate t and heading h; retain the vertex exactly when the whole placed context stays in $[-10, 0]$. An edge performs one literal turn and move and applies the suffix transition, provided its target is retained. This gives 54,718 vertices and 90,158 edges.

For a periodic physical trace, each retained context is a suffix of its true history and therefore lies inside the same strip. After the finite lookback has been supplied, the context depends only on phase, so the trace lifts to a closed graph walk. Repeated vertices are allowed. This coverage statement, rather than a converse realization claim for all graph paths, is what the upper bound uses.

Set

$$\alpha_t = -2(-1)^t (t + 5)^2, \tag{4}$$

whose values from -10 through zero are

$$(-50, 32, -18, 8, -2, 0, -2, 8, -18, 32, -50).$$

The integer recipe starts all vertex potentials at zero and repeatedly raises y_w to $\max(y_w, y_v + 1 + \alpha_t \varepsilon)$ along edges. A guard at 180 makes this reconstruction a finite procedure; every edge is checked afresh at termination. The retained certificate passes all inequalities

$$y_v - y_w + \alpha_t \varepsilon \le -1, \qquad \varepsilon = 1 \text{ (R)}, \quad -1 \text{ (L)}. \tag{5}$$

Its maximum potential is 180; the verifier checks each integer edge inequality directly.

Lemma 4.1 (Profile budgets)**.** *The five profiles of Proposition 3.1 have respective all-period bounds* $192, 168, 160, 168, 192$.

Proof. Sum (5) along the actual lifted closed walk. Vertex potentials telescope. Lemma 2.1 gives the signed turn excess o_t at level t, so

$$P \le -\sum_t \alpha_t o_t.$$

Substitution of the five profiles yields the displayed bounds. □

The same graph and potentials certify all profiles. The graph forgets the longitudinal residue only for this necessary upper bound. The exact search in the next section retains both residue classes.

5 Finite residue and count reductions

Sweep the levels from top to bottom. At a class adjacent to the processed part, let I, J be the numbers of arrivals and departures on that side. Number the class events by their stable ranks. Let a count processed-side arrivals on even ranks and $o \in \{0, 1\}$ be the class's odd indicator. Let a_k and b_k indicate arrival from and departure to the processed upper side at stable rank k. The alternating turns give

$$b_k = a_k \oplus (k \bmod 2) \oplus (t \bmod 2).$$

For even t, departure-ones count as $a+(\lfloor m/2 \rfloor-(I-a))$; for odd t, they count as $(\lceil m/2 \rceil-a)+(I-a)$. Solving for m gives the two cases below. A class length m is count-feasible exactly when

$$m = \begin{cases} 2(I+J-2a)+o, & t \text{ even}, \\ 2(J-I+2a)-o, & t \text{ odd}, \end{cases} \quad m \geq 0, \quad 0 \leq a \leq \lceil m/2 \rceil, \quad 0 \leq I-a \leq \lfloor m/2 \rfloor. \tag{6}$$

Indeed the two rank parities provide precisely these available arrival slots; the local turn rule fixes departures. This is exact local count feasibility, not sufficient global chronological realizability.

For a frontier (b_0, b_1, c_0, c_1) of outgoing and incoming edge counts, the new row sees the following counts. At even levels arrivals are horizontal and switch $r = x \bmod 2$, while departures are vertical and preserve it; at odd levels the roles reverse:

$$(I, J) = \begin{cases} ((b_0, b_1), (c_1, c_0)), & t \text{ odd}, \\ ((b_1, b_0), (c_0, c_1)), & t \text{ even}. \end{cases}$$

If its lengths are m_0, m_1, the next frontier is

$$(m_0 - J_0, m_1 - J_1, m_0 - I_0, m_1 - I_1). \tag{7}$$

Start with $(1, 1, 1, 1)$ at a saturated rail and require the same frontier at the other rail.

Modulo four, the a term disappears from (6); hence the residue frontier is uniquely determined by the wake bits. It can be propagated in either direction. A wrong terminal residue rejects a mask at every period. A normalized physical event also satisfies

$$i \equiv 2r + (t \bmod 2) \pmod 4. \tag{8}$$

This follows directly by alternating vertical and horizontal unit moves. The four phase groups have $P/4$ events. Their count residues must agree, and their common residue fixes P modulo sixteen.

Proposition 5.1 (Finite input inventory)**.** *For $h = -9, -7, -5, -3, -1$, respectively, the modular tests leave $12, 20, 12, 20, 12$ normalized masks. Thus 76 wake assignments remain.*

Proof. Evaluate (6)–(7) modulo four for the 128 masks per profile, and check (8). The supplement regenerates every integer calculation in both directions and checks complementary masks. It also tests all 2304 masks before normalization and the profile parity rejection for the four even empty levels. No numerical optimizer is involved. □

For chronology pruning we also use nonnegative completion tables from (6)–(7). At each frontier retain the least cost of a counted completion. Because future count transitions depend only on the frontier, this is an exact minimum for the relaxation and a safe lower bound for a real completion. The complete table is regenerated from the wake by this finite recurrence.

6 Chronology-exact enumeration and its coverage

In each full class (t, r), list all events in stable order. Their turns are fixed: even ranks right, odd ranks left. Enumerate their incoming sides; literal turning fixes their outgoing sides. In a fixed checker sector an undirected lattice edge has one direction of use. Therefore its ordered source and target occurrences match first-to-first, with no free matching permutation. For class representatives $b_{t,r} = (r, t - r)$ the matching edge has integer voltage δ determined by

$$b_{\text{source}} + \text{outgoing unit vector} = b_{\text{target}} + \delta(2, -2).$$

Lemma 6.1 (Exact event-graph test)**.** *A fully joined event graph represents a finite-support periodic trace exactly when it forms one directed cycle with total voltage one and, along that cycle, the lifted times $j - Pv$ increase strictly with stable rank in every class.*

Proof. A real trace yields precisely such ordered edge matches and one phase cycle. Conversely the voltage identities reconstruct its lattice walk, headings and drift. The stated time order gives the alternating stabilized word of Lemma 2.1, which constructs a finite seed for the infinite repetition. A separate full-field parity calculation verifies every accepted seed. □

The following conditions prune a partial graph while preserving every possible completion. Count tables only lower-bound unbuilt cost. A premature directed cycle cannot later join mandatory unbuilt events. The four phase groups cannot exceed the cap divided by four. Within a fixed open chain, consecutive reads of the same lifted cell must have consecutive stable ranks: a missing rank cannot be inserted into a trajectory segment whose every update is already fixed.

For completeness, the period-dependent time pruning does not assume the final period equals the cap. A partial graph is a union of open directed paths. For an event u, let $c(u)$ be its path, p_u its step offset from the first event, and v_u the accumulated edge voltage along that offset. Let z_c be the unknown phase of the first event on path c. The stabilized time is

$$\tau_u = z_{c(u)} + p_u - Pv_u.$$

Consecutive class ranks require a gap at least four by (8). A simple cycle of these difference inequalities on q chain variables has weight $C + KP$ with integer K. Indeed a required rank step $u \to u'$ gives

$$z_{c(u')} \geq z_{c(u)} + 4 + p_u - p_{u'} + P(v_{u'} - v_u).$$

A simple constraint cycle uses each of its q paths once. On a path with n_c events, the two selected offsets differ by at least $-(n_c - 1)$. Thus $C \geq 4q - \sum_c (n_c - 1) \geq 5q - n$, where n events have been built. If $K \geq 1$ it is positive at every $P \geq n$; if $K = 0$ its sign is constant; if $K < 0$ it decreases with P. Thus a positive constraint cycle at the search cap excludes every admissible smaller period. All operations use exact integers.

No pruning assumes bounded physical memory or a preselected seed box. Only completed traversals are exclusions. A time or node limit is explicitly inconclusive.

6.1 A disjoint partition of the final difficult case

One case, $h = -1$ and mask 187, requires a larger traversal. Its phase residues are all two, so $P \equiv 8 \pmod{16}$; the 192 upper bound therefore reduces to a search cap of 184.

The top-down search deterministically enumerates the 673 partial graphs at the completed-layer frontier preceding row -5. Shard j of eight follows exactly the ordinals congruent to j modulo

eight. Each complete candidate crosses that frontier once. Every shard uses the same recursion and rollback rules to enumerate this cut list, so their disjoint union covers the entire search. Every shard must finish. Matching cut counts and digests are additional implementation controls, not a replacement for this partition argument.

7 Completed classification and proof of the main theorem

Table 1 gives the completed search totals. These count constructed prefixes, not distinct words or simulated ant updates. The last row includes prefixes repeated before the partition frontier in the eight shards.

Empty level	Period bound	Masks	Visited prefixes	Accepted traces
-9	192	12	56 388 171	none
-7	168	20	10 030 681	none
-5	160	12	2 353 673	none
-3	168	20	23 059 897	standard only
-1	192	12	299 233 450	none

Table 1: Completed exact enumeration. All caps are justified before the search; there is no period cutoff in the main theorem. The final difficult case uses cap 184 after its independently checked phase congruence.

All 76 canonical masks are covered by 83 completed native runs, counting the eight shards separately. The total is 391 065 872 prefix visits. The sole survivor is mask 163 with empty level -3. Its complementary mask 92 is the same geometric trace with the other top event chosen as origin.

The complete odd-class set is consequently forced to be

$$O_{\mathrm{std}} = \{(-10,0),(-10,1),(-9,1),(-8,1),(-7,0),(-6,0),\\ (-5,0),(-4,1),(-2,0),(-1,1),(0,0),(0,1)\}.$$

The normalized word, head east at the origin, is the concatenation of these two lines:

```
RRRLRLLRRLLLLRRLLRRRRLLRRLRLLRLLRLRLLRRRRLLRLLRRRRLL
RRRRLLRLRRRRLRLLLLRRRRLRRLRRRRLLLLRLRRRRLRRRRLLLLRLR
```

It is a cyclic phase of the fixed standard word. A complete seed is

$$B = \{(-7,0),(-7,1),(-6,-3),(-6,-2),(-6,-1),(-6,2),\\ (-5,-2),(-5,0),(-5,1),(-5,2),(-4,-2),(-4,1),\\ (-3,-3),(-3,1),(-2,-3),(-2,-1),(-2,1),(-1,-2),(-1,0)\}.$$

Nineteen is not claimed to be the minimum seed size over other phases.

Proof of Theorem 1.2. Proposition 3.1 gives the five profiles and saturated rails. Complementary-origin normalization and Proposition 5.1 leave the 76 masks. Lemma 4.1 bounds every possible period, and the chronology-exact enumeration covers every candidate below these bounds by Lemma 6.1 and the sound pruning arguments. The completed inventory contains exactly the displayed standard word. Its separate infinite-future verification supplies existence. Therefore $P = 104$ and the wake is standard. □

7.1 A shorter parity route to the same classification

The mixed-wake identity in Appendix B removes three of the five profiles analytically. In the top-rail frame its Q layers are the odd levels: translating by $(0, 1)$ changes the checker convention and leaves the lane labels and order unchanged. If h is the missing layer, then $V = 4$ and $J = (h + 37)/2$. The condition $J + V/2 \equiv 1 \pmod 2$ leaves precisely $h = -7, -3$. It is independent of all eight interior residue choices. For $h = -7$, the twenty retained normalized chronological exclusions at cap 168 apply. The $h = -3$ case is the standard layer-profile theorem, including its complete residue classification. This yields an alternative proof using those two retained subcertificates. The full five-profile computation above remains a complete route with the same conclusion.

More generally, put the full extreme rails at $-2k, 0$ and one odd class at each interior level except $h = -2k + 1 + 2j$. Heading restoration forces k odd. Then $V = k - 1$ and $J = (k + 2)(k - 1)/2 + j$, so the identity requires j odd, equivalently $h \equiv 1 \pmod 4$. This derivation supplies the necessary single-gap parity test at every such width.

8 Evidence and reproducibility

The core theorem is computer-assisted: its logical coverage is a written mathematical argument and its finite claims are checked by integer programs. The supplemental source hashes identify the executed programs and the complete run records identify the finite exclusions.

The computational supplement is the immutable snapshot of `kehao95/langtons-ant-core`, revision `e93252230ea9`. Within `wake_synthesis/`, `sources.json` lists the distributed files and their SHA256 hashes; `evidence/profile_classification/manifest.json` identifies the classification programs and run evidence, and the adjacent `summary.json` and `searches.json.gz` record all 76 cases and 83 jobs. The entrypoint is `wake_synthesis/reproduce.py`, run from the repository root. This snapshot identifies the present thin-wake computation separately from the standard-wake archive [5] and the inverse-synthesis archive [4].

The supplement contains the count routines, both ordered-event traversal directions, the local rank check, the upper-bound construction and complete computational certificates. A regenerated count implementation is cross-checked with the retained one. The small shard controls compare their union with an unsplit positive search. The full replay checks every case, all eight final shards, and the unique finite seed. Single active-cell deletions and a far-ahead blocker are rejected by the independent infinite verifier.

```
python3 wake_synthesis/reproduce.py --jobs 4
python3 wake_synthesis/reproduce.py --quick
```

The second command explicitly skips the 76-case exclusion replay, while checking its upper certificate, count tables and controls. The default is full replay, including the periodic-growth and mixed-wake checks. The option `-extensions-only` replays the arithmetic modules, including the general quadratic character, analytic unit-diagonal regressions and the thin-profile parity reduction, while retaining the classification evidence. Python's standard library and a C++17 compiler are sufficient; no linear optimizer, SAT solver or network access is required. Every exclusion in the classification is supported by a completed traversal.

8.1 Exact all-future seed verification

For a proposed word of period P, let q_i be its phase-i position, d its nonzero drift and B its complete finite initial black set. At epoch $n \geq 0$, the colour read at phase i is

$$c_i(n) = \mathbf{1}_B(q_i + nd) \oplus \bigoplus_{\substack{0\leq j<P,\ s\in\mathbb{Z} \\ q_j - q_i = sd \\ s>0 \text{ or } (s=0,\ j<i)}} \mathbf{1}_{\{n\geq s\}}.$$

Indeed the earlier visit at phase j has epoch $n - s$; the displayed condition is exactly its precedence over epoch n, phase i. The required colour is $\mathbf{1}_{\{w_i=\mathrm{L}\}}$. For each i, only finitely many thresholds s and initial-black epochs occur. Probe epoch zero, every threshold, every initial-black epoch, and the epoch immediately after each initial-black epoch. The formula is constant between these probes and beyond the last one, so this finite test proves every future read. The independent implementation is `verification/profile_classification/verify_word.py`, function `forever`, in the pinned supplement.

9 Limitations

The theorem reduces the remaining periodic problem: a nonstandard highway retaining drift $(2, -2)$, growth twelve and width ten must have a thicker interior wake. It remains necessary to determine whether such wakes exist, whether different numerical invariants are possible, and whether all finite seeds eventually become periodic. Those questions are not answered by the present classification. The inverse-synthesis bound and quadratic wake reconstruction apply to non-axis drift. In the moment construction an axis-normal coordinate vanishes, preventing a positive κ; in the quotient graph, axis drift permits boundary-free cycles. These methods therefore leave axis synthesis open. The parity screens are necessary conditions: a passing wake still requires chronological realization and the supplied-word criterion. The finite period cutoff ensures termination of synthesis, rather than a polynomial search-time bound.

A Wake-dependent period bounds and inverse synthesis

The standard-wake classification is one instance of a general inverse problem. Prescribe a non-axis drift d, an initial heading, and a finite set $O \subset \mathbb{Z}^2/\mathbb{Z}d$ of odd classes. We seek a periodic word and finite seed with exactly these data. The wake-dependent bounds [4] determine a finite search range directly from the input wake.

Lemma 2.1 gives $\#\mathrm{R} - \#\mathrm{L} = |O| =: g$. Heading restoration implies $g \equiv 0 \pmod 4$, P even and $d_x + d_y$ even. For $d = (m, -m)$, the complete class label is $(x + y, x \bmod m)$. The checker-sector invariant σ is determined by the supplied pose.

A.1 An integer clock for diagonal budgets

After rotating a diagonal drift parallel to $(1, -1)$, the transverse coordinate $x+y$ and heading return after each period. A clock using these quantities has a cancelling head correction: its period gain comes from the wake. We recall the unit increment identity of [3] before computing that gain.

Fix an integer c, set $s = x + y - c$, and put

$$f(s) = 3^{|s|} - 1, \qquad \epsilon(h) = \begin{cases} +1, & h = N, E, \\ -1, & h = S, W, \end{cases} \qquad G(s, e) = \begin{cases} \frac{1}{2} f(s), & es \geq 0, \\ -\frac{3}{2} f(s), & es < 0. \end{cases}$$

Use the integer clock of [3] on the complete finite state

$$\Theta_c(X) = -\sigma(X) \sum_{q \in B} \chi(q) f(q_x + q_y - c) + G(p_x + p_y - c, \epsilon(h)). \tag{9}$$

Since $f(s)$ is even, Θ_c is an exactly computable integer.

Lemma A.1 (Integer clock, Ke [3])**.** *For every finite-support state and integer centre c, with e, e' the before and after values of ϵ and $s = p_x + p_y - c$,*

$$\Theta_c(FX) - \Theta_c(X) = 1 + 4f(s)\mathbf{1}_{\{e' \neq e,\ es<0\}} \ \geq 1. \tag{10}$$

Proof. Write $\delta = +1$ on R and -1 on L. Directly checking the two colours at four headings gives $\delta\sigma\chi(p) = ee'$, and the outgoing step gives $s' = s + e'$. Thus the black-field increment is $-ee'f(s)$. Let $G_\pm(s) = G(s, \pm 1)$. The elementary recurrences are

$$G_+(s+1) - G_+(s) = f(s) + 1, \qquad G_-(s-1) - G_-(s) = f(s) + 1,$$
$$G_-(s) - G_+(s) = -2\,\mathrm{sgn}(s) f(s).$$

If $e' = e$ the total gain is one. For $+1 \to -1$ it is $1 + 2f(s) + G_-(s) - G_+(s)$, equal to one for $s \geq 0$ and to $1 + 4f(s)$ for $s < 0$. The other reversal gives the reflected cases. This includes $s = 0$. □

A.2 Diagonal wake budgets and crossing refinement

Rotate a diagonal drift to $d = (m, -m)$, $m > 0$, and use the transformed heading. Let $t = x + y$, let o_t count odd classes at level t, and define

$$A_t = -\sigma(-1)^t o_t, \qquad K_c(O) = \sum_t A_t(3^{|t-c|} - 1). \tag{11}$$

Theorem A.2 (Wake-determined diagonal period)**.** *Every realization of this wake obeys $P \leq K_c(O)$ for every integer c.*

Proof. Over one period the head term of (9) cancels, since both t and the heading return to their initial values. The signed cell toggles give $-\sigma \sum_i \chi(q_i) f(t_i - c)\delta_i$. The weight is constant in each full d-class. Lemma 2.1 makes its net δ count one on odd classes and zero on even classes. The exact period increment is therefore $K_c(O)$. Summing (10) proves the bound. The unchanged part of the full field cancels from the period increment. □

Here is an independent geometric derivation, which also supplies a refinement. The finite projected period is a closed walk with increments ± 1 in t. Let f_t count upward crossings of the cut $(t, t+1)$; it also counts downward crossings. Let m_t count visits to t, and u_t count passages entering and leaving in the increasing direction. Flow conservation equates this with the number of decreasing-through passages. The fixed arrival axis and the two turn choices give

$$m_t = f_{t-1} + f_t = A_t + 4u_t, \qquad 0 \leq u_t \leq \min(f_{t-1}, f_t). \tag{12}$$

For example, at vertical-arrival levels, right turns are through passages and left turns reverse the projection, giving $\#\mathrm{R} - \#\mathrm{L} = 4u_t - m_t$; horizontal arrival reverses that sign. Substituting $u_t \leq f_{t-1}$ into $f_t = A_t + 4u_t - f_{t-1}$ gives the first inequality below; $u_t \leq f_t$ gives the second:

$$f_t \leq 3f_{t-1} + A_t, \qquad f_{t-1} \leq 3f_t + A_t. \tag{13}$$

Let ℓ, h be the extreme wake levels. Outside them $A_t = 0$; starting past the finite support of the projected walk and applying (13) inward forces $f_{\ell-1} = f_h = 0$. Thus the periodic trace lies between the extreme wake levels. An empty wake forces every $f_t = 0$, impossible for a nonempty period.

Starting from these two zero boundary values, set

$$L_{\ell-1} = 0, \quad L_t = 3L_{t-1} + A_t, \qquad R_h = 0, \quad R_t = 3R_{t+1} + A_{t+1}, \quad C_t = \min(L_t, R_t).$$

Then $0 \le f_t \le C_t$ for $\ell \le t < h$. If a C_t is negative, the wake is impossible; otherwise

$$P = 2\sum_t f_t \le 2\sum_t C_t. \tag{14}$$

Moreover $d_x = \sigma \sum_t (-1)^t f_t$: on a cut with $\sigma(-1)^t = 1$, the upward horizontal moves contribute $+f_t$ to x; on the other cuts the downward horizontal moves contribute $-f_t$. If C_+ and C_- sum the corresponding positive and negative cut bounds, then also

$$P \le 4C_+ - 2d_x, \qquad P \le 4C_- + 2d_x. \tag{15}$$

Corollary A.3 (Width and growth bound)**.** *Writing $W = h - \ell$, a diagonal realization has even W and*

$$P \le g(3^{W/2} - 1). \tag{16}$$

Proof. At a transverse extreme every passage reverses the projection. The fixed arrival axis makes all letters in an extremal class equal. Lemma 2.1 therefore forces exactly one R per such class. Both extremes have the same arrival axis, so W is even; this is the usual extremal rigidity argument. Take $c = (\ell + h)/2$ in Theorem A.2, discard negative coefficients and use $\sum_t \max(A_t, 0) \le g$ to obtain (16). □

For the normalized standard wake, $\sigma = -1$ and the coefficients at $t = -10, \dots, 0$ are

$$A = (2, -1, 1, -1, 1, -1, 1, 0, 1, -1, 2).$$

The cut bounds are $C = (2, 5, 16, 47, 142, 145, 48, 16, 5, 2)$, with sum 428. Thus (14) gives $P \le 856$, as does (11) with centre $c = -5$. The chronological constraints in Section 4 strengthen this budget to the cutoff used in the complete classification.

A.3 General non-axis drift by reciprocal moments

For a general drift, use a transverse linear coordinate constant along it. An exponential weight and its reciprocal give complementary period budgets; Cauchy–Schwarz then bounds the number of steps independently of their individual weights.

Rotate the drift to $(a_0, -b_0)$, $a_0, b_0 > 0$. Put $k = \gcd(a_0, b_0)$, $\eta = (b_0/k, a_0/k)$, and choose rational $r > 1$ such that, with $a = r^{\eta_x}$ and $b = r^{\eta_y}$,

$$\kappa = 1 - \frac{2}{a+1} - \frac{2}{b+1} > 0.$$

The choice $r = 4$ always works. Define $w(q) = r^{\eta \cdot q}$, and

$$(\gamma_N, \gamma_E, \gamma_S, \gamma_W) = \left(\frac{2}{b+1}, \frac{2}{a+1}, -\frac{2b}{b+1}, -\frac{2a}{a+1} \right).$$

The one-sided potentials are

$$\Psi_+(X) = -\sigma \sum_{q\in B} \chi(q)w(q) + \gamma_h w(p), \qquad \Psi_-(X) = -\sigma \sum_{q\in B} \frac{\chi(q)}{w(q)} + \frac{\gamma_{h+2}}{w(p)}.$$

Substitution of the current-cell flip gives $\Delta\Psi_+ = \kappa w(p)$ in six heading/colour cases and $(\kappa+4)w(p)$ in S/L and W/R. The half-turn formula gives $\Delta\Psi_- \geq \kappa/w(p)$. These local identities are the one-sided exponential clock construction, included to make the new period argument self-contained.

Define the two finite wake moments, using any representatives of O,

$$A_+ = -\sigma \sum_{q\in O} \chi(q) r^{\eta\cdot q}, \qquad A_- = -\sigma \sum_{q\in O} \chi(q) r^{-\eta\cdot q}.$$

They are independent of the representatives since $\eta \cdot d = 0$ and $d_x + d_y$ is even. The head terms cancel over a period, and Lemma 2.1 gives exact increments A_+ and A_-. Thus both are positive and $\kappa \sum_i w(q_i) \leq A_+$, $\kappa \sum_i w(q_i)^{-1} \leq A_-$. Cauchy–Schwarz yields:

Theorem A.4 (Reciprocal wake bound)**.** *Every realization with the specified non-axis drift and wake satisfies*

$$P^2 \leq \frac{A_+ A_-}{\kappa^2}. \tag{17}$$

All quantities are rational. The integer square root of the floor of the right side is a computable cutoff. Changing the transverse centre scales A_+ and A_- reciprocally, leaving the product unchanged. The non-axis hypothesis ensures $\kappa > 0$.

A.4 All-period synthesis and detour obstruction

Theorem A.5 (Period-unknown inverse synthesis)**.** *Given a non-axis integer drift, an initial heading and a finite odd-class wake, existence of a finite-support periodic realization is decidable. A positive answer can produce its word and an explicit finite seed.*

Proof. First reject odd $d_x + d_y$, growth $|O|$ not divisible by four, or a nonpositive moment A_+ or A_-. Otherwise compute Theorem A.4's cutoff, additionally using Theorem A.2 and (14)–(15) in the diagonal case. A negative crossing bound also rejects the input. Enumerate every nonempty word up to the least derived bound. Check its heading and drift, apply Lemma 2.1, and require the exact odd-class set. A passing word constructs a finite seed; complete finite exhaustion gives a negative answer. Completeness follows from the period bound. □

The all-future verifier of Section 8 independently checks the constructed finite seed. For cutoff B, enumeration visits at most $2^{B+1} - 2$ nonempty binary words; the period bound guarantees termination.

Corollary A.6 (No wake-preserving closed-pose insertion)**.** *Suppose realizable words u, v have the same non-axis drift, and v is obtained by inserting nonempty detours into u, each returning to the same position and heading while leaving the original steps unchanged. Then*

$$A_+(O_v) > A_+(O_u), \qquad A_-(O_v) > A_-(O_u).$$

Thus their signed transverse wake cannot be identical. For diagonal drift, $K_c(O_v) - K_c(O_u) \geq |v| - |u| > 0$ for every integer c.

Proof. Every retained original step has the same position, heading and letter; hence it has the same local clock increment. Every inserted step has strictly positive increment in both one-sided clocks, and at least one in the integer clock. Sum over the words and use the exact wake-increment equalities. □

Thus any such realizable detour requires an additional wake budget.

B Mixed-wake interleaving at arbitrary width

The thin-wake classification uses a particular growth and width. A separate parity identity constrains arbitrary finite odd-class wakes at the same full drift $(2, -2)$, without a period or width cutoff. It tests the transverse order of the classes, in addition to their signed residue counts.

Translate a vertical-arrival phase to the origin. Then even levels $t = x + y$ have vertical arrivals, denoted Q, and odd levels have horizontal arrivals, denoted C. Lanes are $r = x \bmod 2$. Let O be the odd classes, $g = |O|$, and define

$$V = |O \cap Q|, \qquad J(O) = \#\{(q, c) \in (O \cap Q) \times (O \cap C) : t(q) < t(c)\}.$$

Distinct classes on the same layer are counted separately.

Theorem B.1 (Mixed-wake interleaving)**.** *Every finite-support periodic realization of full drift* $(2, -2)$ *satisfies*

$$V \equiv 0 \pmod 2, \qquad J(O) + V/2 \equiv 1 \pmod 2.$$

The condition can be evaluated in $O(g \log g)$ *integer operations by sorting only the occupied odd classes, irrespective of transverse gap sizes.*

B.1 Signed residues and the rank factorization

We use the stable-class and signed-residue arguments of Jillhewar [2, Section 7, "Signed mod-four wake-residue identity", and the proof of "Even-diagonal growth four is impossible", equation labelled `eq:diagK`]. The supplemental `proof/periodic-growth-closure.md` supplies an attributed derivation, including an explicit mod-four potential. For a function $f : \mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$, choose $S(0) = 0$ and $S(x + 1) - S(x) = f(x) + f(x + 1)$. With $\chi = (-1)^{x+y}$, put

$$\begin{aligned} F_N &= -\chi S(x), & F_E &= \chi(S(x) - f(x)), \\ F_S &= -\chi(S(x) + 2f(x)), & F_W &= \chi(S(x) + f(x)). \end{aligned}$$

The complete-state quantity $\sum_{(x,y)\in B} \chi(x, y)f(x) + F_h(p)$ is conserved modulo four, as direct substitution in the eight heading/colour cases verifies. For a residue indicator modulo a nonzero horizontal displacement a, its monodromy gives

$$\sum_{q\in O,\ q_x = r \bmod |a|} \chi(q) \equiv 2 \pmod 4.$$

Quarter-turning gives the vertical counterpart. In particular, with $q_r = |O \cap Q \text{ in lane } r|$ and $c_r = |O \cap C \text{ in lane } r|$,

$$q_0 - c_0 \equiv q_1 - c_1 \equiv q_0 - c_1 \equiv q_1 - c_0 \equiv 2 \pmod 4.$$

Thus V is even and $V/2 \equiv q_0 \pmod 2$.

The phase successor acts on the P period events. Order those events first by translation class and then by stable rank within each class. The actual map sends phase i to phase $i+1 \pmod P$, so it is one P-cycle. To factor it in the class order, match incoming rank to outgoing rank at each local turn, sort outgoing events by their port bits, transfer each directed edge block in its preserved order, and undo the incoming-port sorting at the target. Multiplying the signs of these permutations gives the relation used below. The same factorization applies to the general drift in Appendix C.

For completeness we evaluate the local signs. At each class, order visits by stabilized rank k, and record positive incoming and outgoing port bits α_k, β_k: the positive ports are N/E at Q, E/N at C. For a binary string with m positions and A ones,

$$\mathrm{inv}(\alpha) \equiv A(m-1) + \sum_k k\alpha_k + \binom{A}{2} \pmod 2.$$

Apply this to both port strings. Directed-edge binomial terms cancel between source and target. If D_v is the odd-class bit and $p_v = \sum_k (\alpha_k + \beta_k) \bmod 2$, the remaining local correction is $p_v(D_v + 1)$ at Q, and $D_v(p_v + 1)$ at C. Denote its sum by $\mathcal{C}$, and the edge-block permutation sign by $\mathcal{E}$. Stable order within each directed edge block is preserved, since its successor time changes by the fixed amount $1 - P$ voltage. The phase successor is one even-length cycle. Its factorization consequently gives

$$K := \sum_{\text{L events}} \mathrm{rank} \equiv 1 + \mathcal{E} + \mathcal{C} \pmod 2.$$

For an actual stable class, L has odd rank. Also $4 \mid P$ and $4 \mid g$, so $K \equiv (P-g)/2 \equiv 0 \pmod 2$. Hence $\mathcal{E} + \mathcal{C} = 1$. This factorization is inherited; the following evaluation for general mixed wakes is the extension.

B.2 A Boolean identity with a telescoping boundary term

All arithmetic in this subsection is over $\mathbb{F}_2$. Write $Q_{j,r} = D_{(2j,r)}$, $C_{j,r} = D_{(2j+1,r)}$. Let e, w count E/W outgoing edges at Q modulo two, and n, s count N/S outgoing edges at C. Local balance gives, as two-vector identities,

$$w_j = e_j + Q_j, \quad s_j = n_j + C_j, \quad n_j = n_{j-1} + Q_j + C_j, \quad e_j = e_{j-1} + Q_j + \mathrm{swap}(C_{j-1}).$$

All counts vanish below and above the finite range of one period. We prove the closed identity

$$\mathcal{E} + \mathcal{C} = J(O) + \binom{g}{2} + q_0. \tag{*}$$

Encode a layer's outgoing parities in the order negative lane zero, positive lane zero, negative lane one, positive lane one. Its internal inversion contributions are

$$S_Q(b) = b_0b_2 + b_1b_2 + b_1b_3, \qquad S_C(b) = b_1b_2.$$

The adjacent-layer term is $A(a,b) = (a_1 + a_3)(b_0 + b_2)$. At separation two the terms are

$$B_Q(a,b) = a_1(b_0 + b_2) + a_3b_2, \qquad B_C(a,b) = a_1b_0 + a_3(b_0 + b_2).$$

Larger separations give no inversions. These expressions follow by ordering the directed targets: Q edges swap lane and C edges preserve it.

Before a new pair of layers, use state (e_0, e_1, c_0, c_1, z), where z is the accumulated Q population parity. The previous N parities are $n_0 = e_1 + z + c_0$, $n_1 = e_0 + z + c_1$. For new bits (Q_0, Q_1, C_0, C_1), the next state is $(e_0', e_1', C_0, C_1, z')$, where

$$e_0' = e_0 + Q_0 + c_1, \quad e_1' = e_1 + Q_1 + c_0, \quad n_r' = n_r + Q_r + C_r, \quad z' = z + Q_0 + Q_1.$$

Set

$$\begin{aligned} a &= (0, e_0, 0, e_1), & b &= (n_0 + c_0, n_0, n_1 + c_1, n_1), \\ u &= (e_0' + Q_0, e_0', e_1' + Q_1, e_1'), & v &= (n_0' + C_0, n_0', n_1' + C_1, n_1'). \end{aligned}$$

Then

$$\begin{aligned} \Delta\mathcal{E} &= S_Q(u) + S_C(v) + A(b, u) + B_Q(a, u) + A(u, v) + B_C(b, v), \\ \Delta\mathcal{C} &= (n_0 + e_0')(1 + Q_0) + (n_1 + e_1')(1 + Q_1) \\ &\quad + C_0(1 + e_1' + n_0') + C_1(1 + e_0' + n_1'), \\ \Delta J &= z'(C_0 + C_1). \end{aligned}$$

For $d = (Q_0, Q_1, C_0, C_1)$, the population correction is

$$\Delta\binom{g}{2} = (n_0 + n_1)\sum_i d_i + \sum_{i<j} d_i d_j.$$

The explicit boundary polynomial

$$H = e_0 + e_1 + z + e_0c_1 + e_1c_0 + e_1c_1 + e_1z + c_0c_1 + c_0z$$

satisfies

$$H(\text{old}) + H(\text{new}) = \Delta\mathcal{E} + \Delta\mathcal{C} + \Delta J + \Delta\binom{g}{2} + Q_0.$$

Substitution with $x^2 = x$ proves the identity in nine Boolean variables; the checker both expands the polynomial and tests all 512 assignments. Summing over any finite number of layer pairs telescopes H. Empty boundary padding makes it zero at both ends, giving (*). Since $4 \mid g$ and $q_0 \equiv V/2$, comparison with $\mathcal{E} + \mathcal{C} = 1$ proves Theorem B.1.

B.3 Growth-eight wakes with two full extreme layers

Suppose $g = 8$ and both residue classes at each transverse extreme are odd. Extreme visits are singleton right turns, hence on C layers. They supply four C classes. Thus V is even and at most four. If $V = 0$, the new parity is zero. If $V = 4$, all C classes are at the extremes; each Q lies below exactly two of them, so $J = 8$ and $J + V/2$ is again even. Therefore $V = 2$, with lane populations $(q_0, q_1) = (1, 1)$, $(c_0, c_1) = (3, 3)$. The x/y residue matrix is $\left[\begin{smallmatrix} 1 & 3 \\ 3 & 1 \end{smallmatrix}\right]$.

If the four internal odd classes have distinct levels, their increasing Q/C sequence must have an even number of Q-before-C pairs. The alternating orders QCQC and CQCQ are excluded for arbitrary gaps and lane choices. QQCC, QCCQ, CQQC and CCQQ pass this particular test; they are candidates requiring the full realization criterion. This consequence does not require a transverse-width bound.

The exact controls include 18,895 geometric heading-restoring words, 65,536 finite layer subsets, all 104 standard phases, and an independent edge-permutation computation. A separate bounded screen at widths 10,12,14 excludes respectively 3,058, 11,374 and 33,320 assignments passing the signed residues and checker-mixing filters. Those finite tables illustrate the screen; the telescoping identity proves it at every width. The complete periodic-growth supplement also retains attributed small-drift exclusions and the regenerated unit-diagonal certificate, keeping their ownership and logical dependencies explicit.

C Quadratic wake tests for non-axis drift

The rule is white/right or black/left, followed by toggling the current site and moving one edge. The black field is a complete finite subset of $\mathbb{Z}^2$. Suppose a nonempty turn word of length P repeats forever, restores the heading, and has nonzero full displacement d. Finite preludes and genuinely unread debris are allowed. We do not replace d by its primitive direction.

The stable-order and port-factorization method of [2] lets us eliminate the unknown event graph and test the wake directly. Assume $d = (a, b)$, $ab \neq 0$. A half-turn makes $a > 0$; heading restoration implies $a + b$ even. Let O consist of full d-classes with odd phase multiplicity and write $g = |O|$. Canonical representatives are

$$\pi_d(x, y) = (r, y - b\lfloor x/a \rfloor), \qquad r = x \bmod a, \quad 0 \leq r < a.$$

Checker parity $c(r, y) = (r + y) \bmod 2$ is well-defined on the quotient. Initially use $Q = \{c = 0\}$ for vertical-arrival vertices and $C = \{c = 1\}$ for horizontal arrivals. Lemma C.4 removes dependence on this convention.

C.1 The graph reconstructed from a wake

Suppose that every horizontal and vertical quotient component contains an even number of O vertices. On each vertical line $r = x \bmod a$, sort the canonical y coordinates and pair consecutive endpoints by finite intervals. Horizontal components have representative row $Y = y \bmod |b|$. Express a point there using

$$k = (y - Y)/b, \qquad X = x - ka,$$

then sort and pair the X coordinates. Let $G_d(O)$ be the union of these horizontal and vertical interval edges. Orient horizontal edges from Q to C and vertical edges from C to Q.

Order vertices by (y, r) and ports negative before positive; positive means movement E or N. An edge has source and target keys $(\text{vertex}, \text{port})$. Let $E_d(O)$ count, modulo two, edge pairs whose source and target orders disagree. Let $N_+(O)$ count positive retained edges, and put $C(O) = |O \cap C|$. Define

$$\mathfrak{q}_d(O) = E_d(O) + N_+(O) + C(O) \pmod 2, \tag{18}$$

$$\Omega_d(O) = \sum_{(r,y)\in O} \lfloor y/|b| \rfloor \pmod 2. \tag{19}$$

The finite graph may be enormous, but Section C.4 does not construct its individual edges.

Theorem C.1 (Wake obstruction)**.** *Every finite-support periodic realization with non-axis full drift satisfies*

$$\boxed{\Omega_d(O) = 1, \qquad \mathfrak{q}_d(O) = 1 + a \pmod 2.}$$

There is no restriction on period, transverse width, growth, retreat, seed size, or wake shape. The characters are independent of representatives, translation, phase origin and checker/heading-axis convention; no heading is a screen input.

Theorem C.2 (Quadratic structure)**.** *On the $\mathbb{F}_2$ vector space of finite indicators with even sums in every horizontal and vertical quotient component, $\mathfrak{q}_d$ is a quadratic form. In particular $\mathfrak{q}_d(A + B) + \mathfrak{q}_d(A) + \mathfrak{q}_d(B)$ is bilinear. Addition is symmetric difference of indicators, not superposition of physical trajectories.*

Theorem C.3 (Compressed complexity)**.** *Given d and explicit representatives of O in binary, signed-residue prefilters followed by Theorem C.1 are computable in polynomial bit time. After the density filter, at most $R = g(a+1) \le g(g/2+1)$ arithmetic edge runs suffice, and the evaluator uses $O(g^4 \log(g+1))$ integer arithmetic operations on $O(B + \log(g+1))$-bit integers, where B bounds coordinate and drift bit lengths. Passing means only* `NOT_EXCLUDED`*, not realizability.*

C.2 Boundary reconstruction and winding

In a full d-class write phase positions $q_i = b_v + k_i d$. A sufficiently distant cell $b_v + nd$ is initially white and eventually has all the phase visits, at times $nP + i - Pk_i$. The necessary stable-class condition of [2] therefore orders visits by $\tau_i = i - Pk_i$ and gives letters $R, L, R, L, \ldots$. The signed excess in a class is its odd indicator $D_v \in \{0, 1\}$. Thus $g = \sum D_v$, and heading restoration gives $4 \mid g$.

The checker/heading-axis invariant assigns incoming axes alternately to the two checker colours. Every visit uses one horizontal and one vertical edge. Modulo two, the horizontal incident-edge boundary and the vertical incident-edge boundary are consequently both D.

Because $b \neq 0$, the horizontal quotient graph is a union of infinite paths: no nonzero horizontal vector is in $\mathbb{Z}d$. The analogous fact holds vertically because $a \neq 0$. On an infinite path, a finite edge set with prescribed boundary is unique: its indicator is the cumulative endpoint parity, hence exactly the intervals pairing consecutive endpoints. A nonempty finite boundary-free edge set on a line cannot exist. Therefore the actual edge-count parities are precisely $G_d(O)$, whatever their unknown even parts may be.

This argument explains the domain restriction. For axis drift, one direction has quotient cycles, and a boundary-free cycle can be added to parity data. See Section 9 for the resulting axis-scope boundary.

The actual period has signed seam crossing number one across the cut from x residue $a-1$ to 0. For a horizontal interval from X_L to X_R, its seam crossing parity is $\lfloor X_R/a \rfloor - \lfloor X_L/a \rfloor$. Summing over paired intervals equals the sum of these floors over all endpoints modulo two. A canonical endpoint (r, y) has $X = r - ak$, with $k = (y - (y \bmod |b|))/b$, so $\lfloor X/a \rfloor = -k$ has parity $\lfloor y/|b| \rfloor$. This proves (19) and $\Omega_d = 1$. On arbitrary representatives the same weight is $\lfloor x/a \rfloor + \lfloor y/|b| \rfloor \pmod 2$; adding d changes it by the even integer $1 + \operatorname{sign}(b)$.

C.3 The one-tour sign and its wake reduction

At a vertex order its m events by stable rank $k = 0, \ldots, m-1$. Let α_k, β_k be positive incoming and outgoing port bits, with totals A, B, and put $p_v = A + B \pmod 2$. At Q, a left turn means unequal bits; at C, it means equal bits. For any binary list,

$$\operatorname{inv}(\alpha) = A(m-1) + \sum_k k\alpha_k + \binom{A}{2} \pmod 2. \tag{20}$$

Indeed, each one at position k sees $m-1-k$ later positions, from which its later ones are subtracted. Summing the latter gives $\binom{A}{2}$.

Put $K = \sum_{L \text{ events}} \text{stable rank} \pmod 2$. Equation (20) applied to both port lists leaves local corrections $p_v(m-1)$ at Q and $p_v(m-1) + \binom{m}{2}$ at C, besides port inversions and positive-edge binomials. At Q, $m \equiv D_v \pmod 2$. At C, counting unequal bits gives

$$D_v = 2(A+B) - 4\sum_k \alpha_k\beta_k - m, \qquad m \equiv 2p_v - D_v \pmod 4.$$

For $D_v = 0, 1$ the two corrections are respectively $p_v(D_v + 1)$ and $D_v(p_v + 1)$. The positive-edge binomials cancel between source and target. If I is the sum of all incoming/outgoing port-list inversion parities,

$$K = I + L, \qquad L = \sum_Q p_v(D_v + 1) + \sum_C D_v(p_v + 1). \tag{21}$$

Along a fixed directed quotient edge of voltage ν, target stable time is source stable time plus $1 - P\nu$. The same constant applies to every occurrence, including the cyclic seam. Occurrences therefore match first-to-first. Sorting outgoing ports, following edge blocks and unsorting incoming ports factors the phase successor. This successor is *one* even-length P-cycle, of odd sign; hence

$$I + E = 1, \qquad K = 1 + E + L. \tag{22}$$

A block reversal of sizes u, v contributes $uv \pmod 2$. Thus E depends only on edge-count parities and equals $E_d(O)$. This is the general version of the attributed special-drift factorization, not a finite-period extrapolation.

Lemma C.4 (Cancellation and conventions)**.** *For $G_d(O)$ one has*

$$\sum_{v \in O} p_v = 0, \qquad L = N_+(O) + C(O) \pmod 2.$$

The character is unchanged by the total vertex order or by exchanging the checker/heading-axis convention.

Proof. The positive incidence of a paired horizontal interval at its two endpoints has parity $1 + c(\text{left}) + c(\text{right})$. A vertical interval has the same formula. There are $g/2$ intervals of each kind; each endpoint checker bit is counted twice. The total is $g = 0 \pmod 2$. Also $\sum_Q p_v = N_+$, since every positive edge touches exactly one Q vertex. Expanding (21) proves the stated expression for L.

Swapping vertices v, w in the order changes E by $\text{out}(v)\,\text{out}(w) + \text{in}(v)\,\text{in}(w) = 2D_vD_w = 0 \pmod 2$. Exchanging checker conventions reverses all edges and complements the ports. Inversion sign is invariant under permutation inversion; the extra port-block reversals contribute $\sum_v (1 + D_v)p_v = 0$, by endpoint cancellation and $\sum_v p_v = 0$. The edge set has even size, because the sum of outgoing degree parities is g. Thus replacing N_+ by $|G| - N_+$ does not change its parity; neither does replacing C by $g - C$. □

Translations merely relabel vertices and possibly exchange conventions, so $\mathfrak{q}_d$ is translation and phase invariant. Moving a seam does not change winding, because every fibre population is even.

For an actual stable class word, every L has odd rank. Hence $K = \#L = (P - g)/2 \pmod 2$. There are $P/2$ horizontal moves with net displacement a, so $P/2 \equiv a \pmod 2$; since $4 \mid g$, $K \equiv a$. Equations (18), (21), (22) prove Theorem C.1.

The reconstruction map $D \mapsto G_d(D)$ is linear by uniqueness of a finite boundary antiderivative. The inversion parity is quadratic in edge indicators, and N_+, C are linear. This proves Theorem C.2. For $(2, -2)$, the previous evaluation in Appendix B is $\mathfrak{q}_d = J + V/2$ under signed residues, where J counts Q-before-C odd-class pairs in transverse order and $V = |O_Q|$. Thus the earlier formula $J + V/2 = 1$ is recovered.

C.4 Compressed evaluation

The signed residue theorem [2, Section 7, "Signed mod-four wake-residue identity"] gives, in every x residue modulo a and every y residue modulo $|b|$, signed odd-class sum 2 (mod 4), with sign $(-1)^{x+y}$. In particular $g \geq 2\max(a, |b|)$. Check positivity and $4 \mid g$, then density *before* allocating residue-sized arrays; rejected inputs need no graph construction. Both drift magnitudes are then at most $g/2$. Signed sums also enforce even fibre populations.

An arithmetic edge run has n edges with source and target keys

$$(s_y + hi, s_x, p), \quad (t_y + hi, t_x, p), \qquad 0 \leq i < n, \quad h > 0. \tag{23}$$

A vertical interval splits by bottom-site parity into at most two step-two runs. For a horizontal interval on representative row Y, split its left positions $X = r + ak$ by r, and, for odd a, by k mod 2. Canonical heights change by $-b$ per k. Reverse the index if necessary to obtain positive step $|b|$ or $2|b|$ and constant port bits. There are at most $2a$ runs per horizontal interval, so $R \leq g(a+1)$. All runs have disjoint edges.

A run has no internal inversions. Between runs A, B, set

$$z = s_A + h_A i - s_B - h_B j, \qquad \delta = (t_A - s_A) - (t_B - s_B).$$

Let $s = 0$ if the source minor key (x_A, p_A) precedes (x_B, p_B), otherwise $s = -1$. Define $u = -\delta$ plus 0 or -1 by the analogous target comparison. The edge pair is inverted exactly when

$$\min(s, u) < z \leq \max(s, u). \tag{24}$$

Two cumulative pair counts suffice. For $ui - vj \leq T$, $0 \leq i < n$, $0 \leq j < m$, put

$$\ell = \mathrm{clip}(\lceil -T/v \rceil, 0, m), \qquad h = \mathrm{clip}(\lceil (un - T)/v \rceil, 0, m).$$

The exact number of pairs is

$$(m - h)n + (h - \ell) + \sum_{j=0}^{h-\ell-1} \left\lfloor \frac{vj + T + v\ell}{u} \right\rfloor. \tag{25}$$

There are no eligible i below ℓ, all n are eligible above h, and the middle contributes $\lfloor (T + vj)/u \rfloor + 1$.

The floor sum $S(n, m, a, b) = \sum_{i=0}^{n-1} \lfloor (ai + b)/m \rfloor$ uses Euclidean lattice transposition. Remove quotient parts of a, b, contributing $\lfloor a/m \rfloor n(n-1)/2 + \lfloor b/m \rfloor n$. For $0 \leq a, b < m$ the remainder is zero if $an + b < m$; otherwise, with $n' = \lfloor (an + b)/m \rfloor$ and $b' = (an + b) \bmod m$,

$$S(n, m, a, b) = S(n', a, m, b'). \tag{26}$$

At positive height j there are $n - \lceil (mj - b)/a \rceil$ eligible indices. Reindex heights from n' downwards, using $mn' - b = an - b'$, to obtain the right side of (26). Division first handles negative offsets exactly. Euclidean reduction needs logarithmically many stages.

All run-pair counts have polynomial bit length even for runs of 10^{100} edges. Summing positive run lengths gives N_+; C and Ω_d use only the endpoints. The run bound, R^2 run pairs, and (25)–(26) prove Theorem C.3. A passing wake proceeds to the chronology-exact inverse synthesis of Appendix A.

C.5 An unbounded growth-eight exclusion

Let $d = (3, 1)$ and

$$O_* = \{(0, 1), (0, 3), (1, -2), (1, -1), \\ (1, 0), (1, 2), (2, 0), (2, 2)\}.$$

For $n \geq 1$, put $\lambda = 12n + 1$ and let O_n be the classes of λq, $q \in O_*$. They are eight distinct classes with canonical representatives

$$(r, (12y - 4r)n + y), \qquad (r, y) \in O_*.$$

Their signed x-fibre sums are $(-2, -2, 2)$ and their sole y-fibre sum is -2. They have growth eight, pass density, and have winding $\sum \text{canonical } y = 28n + 5 \equiv 1 \pmod 2$.

Corollary C.5. *No finite-support periodic highway of drift $(3, 1)$ has wake O_n, for any integer $n \geq 1$ and any period. Their algebraic transverse span, measured by $T(x, y) = 3y - x$, is $16(12n + 1)$ and is unbounded.*

Proof. The affine runs in Appendix C.7 give

$$|G_d(O_n)| = 192n + 16, \quad N_+(O_n) = 96n + 9, \quad \#\text{inversions} = 400n + 19, \quad C(O_n) = 5.$$

Thus $\mathfrak{q}_d(O_n) = 496n + 33 \equiv 1$, whereas Theorem C.1 requires $1 + a \equiv 0$. For completeness, the calculation is an all-parameter integer identity, not an interpolation. If $L < R$ are horizontal representative-row endpoints before dilation, the index bounds at residue r are

$$k_{\min} = 4Ln + \lceil (L - r)/3 \rceil, \qquad k_{\max} = 4Rn + \lfloor (R - 1 - r)/3 \rfloor.$$

Splitting parity gives 32 runs of step two and positive affine lengths on $n \geq 1$. For a parity-compatible inversion difference z, set $k = (z - s_A + s_B)/2$. Its count is exactly

$$\max\{0, \min(n_B, n_A - k) - \max(0, -k)\}. \tag{27}$$

Every minimum/maximum branch is verified uniformly: an affine expression $An + B$ is nonnegative for every $n \geq 1$ exactly when $A \geq 0$ and $A + B \geq 0$. Summing the 288 parity-compatible terms among 496 run pairs gives $400n + 19$. The supplement implements this symbolic arithmetic and fails rather than extrapolating if a comparison changes sign on the parameter domain. □

The unscaled O_* also fails: its counts are 16 retained edges, 9 positive edges, 23 inversions and $C = 5$. For contrast,

$$\{(0, 1), (0, 5), (1, -4), (1, -2), (1, 5), (1, 6), (2, -4), (2, 4)\}$$

has the same signed fibre populations and winding, but $\mathfrak{q}_d = 0$. It satisfies this necessary parity condition. Thus the character strengthens the specified signed-residue, density and winding conditions. Physical realization additionally requires chronological compatibility of the visits.

C.6 Exact computational checks

The proof and complexity theorem do not depend on bounded experiments. Exact affine algebra establishes the entire family parameter range. Independent implementations check 10,000 floor sums and 10,000 pair-count instances against literal sums, 4,000 individual-edge reconstructions for eight drift types, vertex-order and checker-convention independence, and quadratic polarization. All 8,192 closed masks in an eight-layer/two-lane window reproduce in Appendix B.

An independent C++ vector-turn implementation enumerates all 1,398,100 binary words of even length at most 20 and checks the unspecialized port factorization on all 456,120 non-axis heading-restoring words. The 84 words with class balances in $\{0, 1\}$ are additionally cross-checked against the wake-only formula. These geometric words are not asserted physically realizable. All 104 standard-highway phases, four rotations and powers one through four provide 1,664 positive controls using their *full* displacements. Representatives with large coordinates and $n = 10^{100}$ are checked without traversing their gaps; the latter still uses 32 runs. Exact reports and source manifests are in the supplement.

C.7 Affine edge-run certificate

All runs have step two and index $i = 0, \dots, \ell - 1$. A row gives length ℓ, source $(s_x, s_y + 2i)$, target $(t_x, s_y + \delta + 2i)$ and port p. These are edges of the parity graph, not time intervals of a proposed ant orbit. The row order is only an enumeration of runs; inversions use vertex order (y, r). Formula (27) on these rows yields the counts in Corollary C.5 for the whole domain $n \geq 1$.

Run	ℓ	s_x	s_y	t_x	δ	p
1	$8n+1$	0	$20n+2$	1	0	1
2	$8n+1$	1	$20n+3$	0	0	0
3	$8n$	2	$20n+4$	1	0	0
4	$8n+1$	1	$20n+3$	2	0	1
5	$8n$	2	$20n+4$	0	-1	1
6	$8n+1$	0	$20n+2$	2	1	0
7	$2n$	0	$12n+2$	1	0	1
8	$2n$	1	$12n+3$	0	0	0
9	$2n$	2	$12n+2$	1	0	0
10	$2n$	1	$12n+3$	2	0	1
11	$2n+1$	2	$12n+2$	0	-1	1
12	$2n$	0	$12n+2$	2	1	0
13	$2n$	0	$-8n$	1	0	1
14	$2n$	1	$-8n+1$	0	0	0
15	$2n+1$	2	$-8n$	1	0	0
16	$2n$	1	$-8n+1$	2	0	1
17	$2n$	2	$-8n+2$	0	-1	1
18	$2n$	0	$-8n$	2	1	0
19	$6n+1$	0	$-28n-2$	1	0	1
20	$6n$	1	$-28n-1$	0	0	0
21	$6n$	2	$-28n$	1	0	0
22	$6n+1$	1	$-28n-1$	2	0	1
23	$6n$	2	$-28n$	0	-1	1
24	$6n+1$	0	$-28n-2$	2	1	0

Run	ℓ	s_x	s_y	t_x	δ	p
25	$12n+1$	0	$12n+3$	0	-1	0
26	$12n+1$	0	$12n+1$	0	1	1
27	$6n+1$	1	$-28n-2$	1	1	1
28	$6n$	1	$-28n$	1	-1	0
29	$12n+1$	1	$-4n$	1	1	1
30	$12n+1$	1	$-4n+2$	1	-1	0
31	$12n+1$	2	$-8n+1$	2	-1	0
32	$12n+1$	2	$-8n+1$	2	1	1

D Analytic exclusion of the unit diagonal

The classical rule is: read white/right or black/left, turn, toggle the current cell, and move one edge on the entire square lattice. The initial black set is finite. Suppose a nonempty word u repeats forever, restores the heading, and has nonzero full displacement d. Set $P = |u|$ and $g = \#R - \#L$. The period need not be least; arbitrary unread finite debris is allowed.

Theorem D.1. *No such periodic highway has full displacement* $(1,-1)$ *or any quarter-turn rotation of this vector, regardless of period, width, growth, or finite seed.*

The full vector is essential: standard P104 has displacement $(2,-2)$ and is not excluded because its primitive direction is $(1,-1)$.

We prove the exclusion by forced layer words and a binary interleaving contradiction. This supplies an analytic alternative to the retained finite certificate.

D.1 Stable ranks and the singleton extreme

Write the phase positions as q_i, $0 \le i < P$. For $d = (1,-1)$, exactly one full translation class lies on each level $t = x + y$. Choose $b_t = (0,t)$ and write $q_i = b_t + a_i d$. At a sufficiently far cell $b_t + nd$, every phase is present, the initial colour was white, and the visit times are

$$nP + \tau_i, \qquad \tau_i = i - Pa_i.$$

Consequently each class, in increasing *stable rank* k, turns $R, L, R, L, \ldots$. This is the necessary part of the finite-support criterion of [2]. A fixed path of ℓ moves and displacement δd changes these stable times by $\ell - P\delta$.

The invariant σ is $(-1)^{x+y}$ for vertical incoming headings and its negative for horizontal incoming headings. Every step changes both factors. At a transverse extreme, the incoming axis and the inward-neighbor condition fix both directions of a visit. All turns of that class are identical, so stable alternation forces one singleton R event. Choose the top event T as a sufficiently late phase origin and translate it to $(0,0)$. It arrives east, leaves south, and gives $\sigma = -1$. The bottom arrives west and leaves north and must also have even transverse level. The width W is therefore positive and even. The new phase's complete field remains finite.

We will repeatedly use two direct consequences of chronology.

Forced matching. If a directed edge satisfies $b_s + e = b_t + \delta d$, it changes stable time by $1 - P\delta$. All source occurrences and all target occurrences have the same order and equal cardinality. They match first to first, not by an arbitrary permutation. At the period seam, the additional epoch cancels in $i - Pa_i$.

Consecutive-return rule. A fixed contiguous path's first return to its starting physical cell must increase that cell's stable rank by exactly one. Translate the path far enough along d for all its reads to be stable. Any intervening read would be another node of the same contiguous path; if none occurs, no stable rank can be skipped. If the endpoints were the same phase event, the path would instead form a closed component. Such a component cannot occur while obligatory deeper events remain: the successor on the P phase indices is one cycle, including when the word is a power of a shorter word.

D.2 The first two interior layers are forced

Build downwards. Let $a_k = 1$ indicate arrival from the already constructed upper side and $b_k = 1$ departure to that side. At even t the incoming headings for bits $1, 0$ are W,E, and at odd t they are S,N. Literal turning gives

$$b_k = a_k \oplus (k \bmod 2) \oplus (t \bmod 2). \tag{28}$$

If the word has length n, write $I = \sum a_k$, $J = \sum b_k$, $o = n \bmod 2$, and let e count arrival-ones at even ranks. Counting departures gives

$$n = \begin{cases} 2(I + J - 2e) + o, & t \text{ even}, \\ 2(J - I + 2e) - o, & t \text{ odd}. \end{cases} \tag{29}$$

The remaining lower cut counts are $n - J$ and $n - I$. They agree, starting with $I = J = 1$ from the top singleton.

Width zero is impossible because a move changes t. For $W = 2$, the extreme singletons force two middle visits, whereas (29) gives $2 = 4e - o$; since $n = 2$ implies $o = 0$, this is impossible. Thus $W \geq 4$, and layers $-1, -2, -3$ are strictly interior, with a deeper visited layer still required.

D.2.1 First layer

Denote rank k at level -1 by A_k. Here $I = J = 1$, so the only feasible arrival words are

$$100, \quad 1000, \quad 001, \quad 0010.$$

Indeed (29) gives $n = 4e - o$, and its single arrival-one lies at even rank zero or two. In either last case, forced matching gives $A_0 \to T \to A_2$, with directions E,S. This travels by d in two updates, so $\tau(A_2) - \tau(A_0) = 2 - P < 0$, contrary to increasing rank. There are at least four distinct phase events, so $P \geq 4$. Only 100 and 1000 remain. Both have the fixed top path $A_2 \to T \to A_0$.

D.2.2 Second layer

Let B_k denote rank k at level -2. Let $p_0 < p_1 < \cdots$ list its arrival-one ranks and $q_0 < q_1 < \cdots$ its departure-one ranks. Each list has size $c = 2$ or 3, according as A has length three or four. Matching forces

$$B_{q_1} \xrightarrow{\mathrm{N}} A_2 \xrightarrow{\mathrm{E}} T \xrightarrow{\mathrm{S}} A_0 \xrightarrow{\mathrm{W}} B_{p_0}.$$

This is a first physical return, so the consecutive-return rule gives

$$p_0 = q_1 + 1. \tag{30}$$

At this even layer, every even rank belongs to both lists or neither, and every odd rank belongs to exactly one. Before p_0 there is no arrival-one. Thus every even rank there is in neither list, and every odd rank is a departure-one. Equation (30) forces

$$q_0 = 1, \qquad q_1 = 3, \qquad p_0 = 4.$$

The even rank four belongs to both lists, giving $q_2 = 4$. Therefore $c = 3$: A has word 1000. The departure list is exhausted; all later even ranks have arrival bit zero and all later odd ranks arrival bit one. The remaining arrivals must be at five and seven, and rank nine cannot occur. Hence B is exactly one of

$$00001101, \qquad 000011010. \tag{31}$$

No case enumeration is used in this forcing argument.

The fixed upper paths now include

$$B_1 \to A_1 \to B_5,$$
$$B_3 \to A_2 \to T \to A_0 \to B_4 \to A_3 \to B_7.$$

The B ranks departing downwards are $0, 2, 5, 6, 7$ and optionally eight; those arriving from below are $0, 1, 2, 3, 6$ and optionally eight. Only the first five of each list will be used, so the optional last event creates no further case.

D.3 A third-layer interleaving contradiction

Let C_k denote rank k at level -3 and reuse p_j, q_j for its arrival-one and departure-one ranks. There are at least five of each. Ordered matching forces the following paths:

$$C_{q_1} \to B_1 \to A_1 \to B_5 \to C_{p_2},$$
$$C_{q_3} \to B_3 \to A_2 \to T \to A_0 \to B_4 \to A_3 \to B_7 \to C_{p_4}.$$

Their compass words are respectively E,N,W,S and E,N,E,S,W,N,W,S. Both are first returns to the starting physical cell, with no other third-layer read between. Thus

$$p_2 = q_1 + 1, \qquad p_4 = q_3 + 1. \tag{32}$$

Two bridges, however, are forbidden. If $p_1 = q_1$, then

$$B_2 \to C_{p_1=q_1} \to B_1 \to A_1 \to B_5$$

has compass word S,E,N,W: its first physical return skips from rank two to five on B. If $p_4 = q_4$, then

$$B_4 \to A_3 \to B_7 \to C_{p_4=q_4} \to B_6$$

has compass word N,W,S,E: its first return skips from rank four to six. Both violate consecutivity, so

$$p_1 \neq q_1, \qquad p_4 \neq q_4. \tag{33}$$

Lemma D.2 (Binary interleaving)**.** *Let p and q be increasing lists of indices in $\{0, \dots, n-1\}$. Suppose every even index in this interval belongs to exactly one list, and every odd index to both or neither. If $p_2 = q_1 + 1$ and $p_4 = q_3 + 1$, then $p_1 = q_1$ or $p_4 = q_4$.*

Proof. If q_1 is odd, it also lies in p, and the next index is p_2, so $q_1 = p_1$. Otherwise q_1 is even and p_2 is odd, giving $p_2 = q_2$. If q_3 is even, p_4 is odd and gives $p_4 = q_4$. Otherwise q_3 is odd and is p_3. Avoiding both asserted equalities therefore gives $p_2 = q_2$ and $p_3 = q_3$, two distinct odd indices that are consecutive in both lists. No list member can lie strictly between them. But an even integer lies between distinct odd integers, and it must belong to exactly one list, a contradiction. □

At odd layer -3, equation (28) is precisely the list condition of Lemma D.2. Thus (32) contradicts (33).

Proof of Theorem D.1. Widths zero, odd widths, and width two were excluded directly. At every other width the first two interior words are forced by Section D.2, and their third-layer continuation is impossible by Section D.3. Quarter-turn rotation preserves the rule and finite support. No finite enumeration is a premise. □

D.4 Growth consequence and attribution

Jillhewar's results [2, Section 7, signed-residue theorem and the axis/even-diagonal growth-four theorems] give $g \in 4\mathbb{Z}_{>0}$ and $g \geq 2\max(|d_x|, |d_y|)$. For $g = 4$, both coordinates have absolute value at most two. Since heading restoration makes P even, $d_x + d_y$ is even. Up to quarter-turns, the only nonzero drift types are $(1, -1)$, $(2, 0)$ and $(2, -2)$. Theorem D.1 excludes the first; the cited all-period axis and even-diagonal growth-four theorems exclude the others. This recovers $g \geq 8$. The supplemental periodic-growth proof provides an attributed reconstruction of those external dependencies. Neither these earlier algebraic results nor the growth corollary is claimed as new here. The analytic unit-diagonal proof uses only the preceding stable-rank and interleaving arguments and does not rely on those other exclusions.

D.5 Independent computational checks

The supplement's `reproduce.py` reruns a complete-row Python enumerator, an independently written C++ vector-geometry enumerator, and witness audits. They agree on 4/124/12 756 local candidates and 2/2/0 survivors. A separate implementation validates all 12 880 rejected branches, 32 766 short binary words, 32 small count domains, eight compass port cases, two corrupt-witness negative controls, and 936 genuine P104 prefix controls retaining the full $(2, -2)$ quotient. An additional analytic audit verifies the displayed first return paths, the forced words (31), and the interleaving implication on every candidate in the finite enumeration. These checks corroborate the local identities; the symbolic forcing and contradiction above establish the all-period exclusion.

Notes on sources

All claim-level citations to reference [2] refer to its pinned August manuscript (revision `76a7e234e577`). The distinct Research Square v1 was posted 30 July 2026, doi:10.21203/rs.3.rs-10516269/v1; its width-two and width-four exclusions do not supply the August width-six and width-eight results.

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